CALCULUS – APPLICATION IN BUSINESS 3

This Modules includes

CALCULUS – APPLICATION IN BUSINESS

Module Learning Objectives:

After studying this Module, the students will be able to understand –

- The meaning of a Function
- The Concept of Limit of a Function
- The Concept of Differentiation of a Function
- The application of differentiation in business situations with special emphasis to the Cost and Revenue functions
- The application of differentiation for optimization

Concept of Calculus and its Application in Business

3.1

FUNCTION

If x and y be two real variables related to some rule, such that corresponding to every value of x within a defined domain or set of values we get a defined value of y, then y is said to be a function of x defined in its domain.

Here the variable x to which we may arbitrarily assign different values in the given domain is known as *Independent variable* (or argument) and y is called the *Dependent variable* (or function).

Notations : Generally we shall represent functions of x by the symbols f(x), F(x), y(x) etc.

Illustration 1. A man walks at an uniform rate of 5 km per hour. If s indicates the distances and t be the time in hours (from start), then we may write, s = 5t.

Here s and t are both variables, s is dependent if t is independent. Now s is a function of t and the domain (value) of t is $0 \leq t \leq \infty$.

Illustration 2. $y = f\left(x\right) = \frac{x^2}{x}$.

For $x \neq 0$, y = x and for x = 0, y is not known (undefined). Here the domain of *x* is the set of real numbers *except zero*. (refer worked out problem 2 of limit & continuity)

Constant Function : y = f(x) = 7 for all real values of x. Here y has just one value 7 for all values of x.

Single-valued, Multi-valued Function: For a function sometimes it may so happen that for y = f (x), there exists a single value of y for every value of x. This type of function is known as single-valued function.

Illustration 3. $y = f(x) = 2x + 3$

For $x = 1, y = 2.1 + 3 = 2 + 3 = 5.$

$x = 2, y = 2.2 + 3 = 4 + 3 = 7$

If again we get more than one value of *y* for a value of *x*, then *y* is said to be a *multiple-valued* (or multi-valued) function of *x*.

Illustration 4. $y^2 = x$. Here for every x > 0, we find two values of y as $y = \pm\sqrt{x}$

Explicit and Implicit Function: A function is said to be *explicit* when it is expressed directly in terms of the independent variable; otherwise it is implicit.

Illustration 5. $y = x^2 - x + 1$ is an explicit function :

$2x^2 + 3xy + y^2 = 0$ an implicit function.

Parametic Representation of a Function : If the dependent variable x be expressed in terms of a third variable, say t, i.e., y = f (t), x = F (t), then these two relations together give the parametic representation of the function between y & x.

Illustration 6. $y = t^2 + 1$, x = 2t.

Odd and Even Functions : A function f (x) is an odd function of x if f(– x) = – f(x) and is an even function of x if f (– x) = = f(x).

Illustration 7. f(x) = x. Now f(–x) = –x i.e f (– x) = – f (x), so f (x) = x is an odd function of x.

$f(x) = x^2$, $f(-x) = (-x)^2 = x^2 = f(x)$, so $f(x) = x^2$ is an even function of x.

Inverse Function : If from a function y = f(x), we can obtain another function x = F(y), then each function is known as the inverse of the other.

Illustration 8. y = 4x – 3 and $x = \frac{1}{4}(y+3)$ are inverse to each other.

Both are the Functions of single independent variable :

Polynomial Function : A function of the form

$$F(x) = a_0 + a_1x + a_2x^2 + \ldots + a_{n-1}x^{n-1} + a_nx^n,$$

(Where n is a positive integer and $a_0, a_1 \ldots. a_n$ are constants) is known as a polynomial function in x.

For n = 0, f(x) = a_0, a constant function

= 1, f(x) = $a_0 + a_1x$, a linear function in x

= 2, f(x) = $a_0 + a_1x + a_2x^2$, a quadratic function in x.

= 3, f(x) = $a_0 + a_1x + a_2x^2 + a_3x^3$, a cubic function in x.

Rational function : A function that is expressed as the ratio of two polynomials

i.e., $f(x) = \frac{a_0 + a_1x + a_2x^2 + \ldots\ldots + a_nx^n}{b_0 + b_1x + b_2x^2 + \ldots\ldots + b_nx^n}$.

i.e., in the form of $\frac{P(x)}{Q(x)}$

is called a rational function of x, such function exists for denominator ≠ 0.

Illustration 9. $f(x) = \frac{x+2}{x^2 - 4x + 3}$ exists for all values of x, if $x^2 - 4x + 3 \neq 0$. Now for $x^2 - 4x + 3 = 0$ or (x – 1) (x – 3) = 0 or, x = 1, 3 Denominator becomes zero and hence the given function does not exist.

Irrational function : On the contrary, if a function f(x) cannot be represented in this form, it is called an irrational function.

Illustration 10. Functions of the form $\sqrt{x}$ (where x is not a perfect square number)

Algebraic function : A function in the form of a **polynomial** with **finite number of terms** is known as algebraic function.

Illustration 11. $x^2 + 2x - 3, \sqrt{x^2 + 1}$ etc.

Domain and Range of a Function :

The set of values of independent variables x is called the '*Domain*' of the function and the set of corresponding values of f(x) i.e. the dependent variable y is called the '*Range*' of the function.

Illustration 12. For the squared function of y = x^2, we get the ordered pairs (1, 1) (2, 4) (3, 9), … etc. as the values of x and y respectively, The set of values {... –2, –1, 0, 1, 2, 3, …} is the domain of the Independent Variable, where as set of values {0, 1, 4, 9, ….} represents the range of the Dependent one.

Illustration 13. For the following functions find the domain and range.

(i) $f(x)=\dfrac{x^2-4}{x-2},\ x\neq 2$ (ii) $f(x)=\dfrac{3-x}{x-3}$, (iii) $f(x)=\dfrac{2x+1}{(x-1)(x+1)}$

Solution:

(i) $f(0)=\dfrac{-4}{-2}=2,\ f(1)=\dfrac{-3}{-1}=3,\ f(3)=\dfrac{5}{1}=5,\ f(4)=6,\ f(-1)=1$

∴ domain = { .... –2, –1, 0, 1, 3, 4, …..}, range = {1, 2, 3, 5, 6, ….}

= R – {2}, = R – {4}, R = real number,

(ii) $f(0)=\dfrac{3}{-3}=-1,\ f(1)=-1,\ f(2)=-1,\ f(-1)=-1$

domain = { – 1, 0, 1,2, 4, ….}, range = { –1, –1, –1, ….} = {–1}

= R – {3}, where R is a real number

(iii) The function of f(x) exists for x ≠ 1, x ≠ – 1.

∴ domain = R – {– 1, 1}, i.e, all real numbers excluding 1 & (–1)

range = R.

Illustration 14. Find the domain of definition of the function $\dfrac{4x-5}{\sqrt{x^2-7x+12}}$

Solution:

Denominator $=\sqrt{x^2-7x+12}=\sqrt{x^2-4x-3x+12}=\sqrt{x(x-4)-3(x-4)}=\sqrt{(x-3)\,(x-4)}$

From above it can be said that if 3 < x <4 then (x – 3) is positive and (x – 4) is negative.

Thus (x – 3) (x – 4) = (Positive) × (Negative) = Negative

Hence $\sqrt{(x-3)\,(x-4)}=\sqrt{\text{Negative}}$ = Imaginary number. Also for x = 3 & 4, denominator is zero.

Thus given function is not defined for 3 ≤ x ≤ 4. So domain is all real values of x except 3 ≤ x ≤ 4.

Absolute Value : A real number "a" may be either a = 0 or, a > 0 or a < 0. The absolute value (or modulus) of a, denoted by | a | is defined as | a | = a, for a > 0

= – a, for a < 0

Thus | – 4| = – (– 4) = 4, and | 4 | = 4.

Complex No :

A number of the form (a+ib) or (a–ib) [where a & b are real numbers] is called a complex number (where $i=\sqrt{-1}$) The complex number has two parts; a real part & an imaginary part. 'a' is the real part & 'ib' is the imaginary part.

Illustration 15.

If y = x^2 + 4 is a function under consideration then solving for y = 0, we get

$0 = x^2 + 4$

or, $x^2 = -4$

or, $x = \pm\sqrt{-4}$

or, $x = \pm 2\sqrt{-1} = \pm 2i$

The number ± 2i is a complex number whose real part is 0 & imaginary part is ±2

Illustration 16. Given f(x) = $2x^2 - 3x + 1$; find f(2), f(0), f (– 3)

Solution:

$f(2) = 2.2^2 - 3.2 + 1 = 2.4 - 6 + 1 = 8 - 6 + 1 = 3$

$f(0) = 2.0^2 - 3.0 + 1 = 2.0 - 0 + 1 = 0 - 0 + 1 = 1$

$f(-3) = 2(-3)^2 - 3.(-3) + 1 = 2.9 + 9 + 1 = 18 + 9 + 1 = 28.$

Illustration 17. If y = 4x – 1, find the value of y for x = 2. Can y be regarded as a function of x? Also find the domain.

Solution:

For x = 2, y = 4.2 – 1 = 8 – 1 = 7. Again for x = 0, y = – 1 and for x = – 1, y = – 5. So for every value of x in $-\infty < x < \infty$, we find different values of y, So y is a function of x and its domain is -

$-\infty < x < +\infty$.

Illustration 18. If f (x) = x + | x |, find f (3) and f (– 3) and show also they are not equal.

Solution:

f(3) = 3 + | 3 | = 3 + 3 = 6 ; f (– 3) = – 3 + | – 3| = – 3 + 3 = 0.

As 6 ≠ 0, so f(3) ≠ f (– 3).

Note : If f(x) = f(– x) [i.e., f (3) = f (– 3)] then f(x) will be an even function of x.

Illustration 19. Show that $\sqrt{x^2 - 5x + 4}$ is not defined for 1 < x < 4

Solution:

$$\sqrt{x^2 - 5x + 4} = \sqrt{(x-1)(x-4)}$$

Now for any value x > 1, but < 4 the expression becomes imaginary. So the expression is undefined for 1 < x < 4.

Illustration 20. Find the domain of $f(x) = \dfrac{x}{x^2 - 9}$

Solution:

Here f(x) has a unique value except for x = 3, – 3.

For $f(3) = \dfrac{3}{9-9} = \dfrac{3}{0}$ (undefined) and $f(-3) = \dfrac{-3}{9-9} = \dfrac{-3}{0}$ (undefined)

∴ domain of the function f(x) is $-\infty < x < -3$; $-3 < x < 3$ and $3 < x < \infty$.

Illustration 21. Given the function

$f(x) = 5^{-2x} - 1,\ -1 \le x < 0$

$= \dfrac{x^2-2}{x-2},\ 0 \le x < 1$

$= \dfrac{2x}{x^2-1},\ 1 \le x < 3$

Find f (– 1), f(0), f (1/2), f (2).

Solution:

$f(-1) = 5^{-2(-1)} - 1$ (since – 1 lies in the interval $-1 \le x < 0$)

$= 5^2 - 1 = 25 - 1 = 24.$

$x = 0$ and $x = \dfrac{1}{2}$ lie in the interval $0 \le x < 1$,

$$f(0) = \frac{0-2}{0-2} = 1,\quad f\left(\frac{1}{2}\right) = \frac{\frac{1}{4}-2}{\frac{1}{2}-2} = \frac{7}{6}.$$

Now 2 lies in the third interval. So $f(2) = \dfrac{2.2}{2^2-1} = \dfrac{4}{4-1} = \dfrac{4}{3}.$

Illustration 22. If $f(x) = e^{ax+b}$. Prove that $e^b f(x+y) = f(x).f(y)$

Solution:

$e^b f(x+y) = e^b.e^{a(x+y)+b} = e^{b+ax+ay+b}$

$= e^{ax+b}.\ e^{ay+b} = f(x).\ f(y)$. Hence proved.

Illustration 23. If f(x) = x – a, q(x) = x + a then show that

$\{f(x)\}^2 - \{q(x)\}^2 = -2a\ \{f(x) + q(x)\}$

Solution:

L.H.S. $= (x-a)^2 - (x+a)^2 = x^2 + a^2 - 2ax - (x^2 + a^2 + 2ax) = -4ax$

R.H.S. $= -2a\ \{x - a + x + a\} = -2a.\ 2x = -4ax$. So L.H.S = R.H.S (Proved)

LIMIT

Introduction:

Calculus is based, in general, on the idea of limit. At present this idea including its related concepts, continuity to mention, will be discussed.

Some definitions:

(i) **Meaning of "x tends to a".** When the difference |x–a| (i.e., numerical difference between the present value of *x* and *a*) can be made less than any positive quantity, however small, we say x tends to a and is written as $x \to a$.

(ii) **Meaning of "x tends to zero".** When the value of x goes on decreasing numerically and can be made numerically less than any positive quantity, however small, we say x tends to zero and is written as $x \to 0$.

(iii) **Meaning of "x tends to infinity".** When the value of x goes on increasing and can be made greater than any positive quantity, however, large, we say x tends to infinity written as $x \to \propto$.

Neighbourhood or Proximity of a point

Let c be any real number, then any open interval around c is called the neighbourhood of c, e.g.,

$$]\, c - \in, c + \in [, \in > 0$$

Is the neighbourhood of c.

Any neighbourhood from which the point c is excluded is called deleted neighbourhood of c.

Geometrically it means set of those points which are within an infinitely small distance $\in$ from c on either side except for the point c.

(iv) **Limit of a function f (x).** A number l is said to be the limit of f(x) as x→a written as $\lim_{x \to a} f(x) = l$ if

a. The function is defined and single valued in the deleted neighbourhood of a.

b. For every positive number $\in$, however small, there exists a positive number δ (usually depending on $\in$), such

$$|f(x) - l| < \in$$

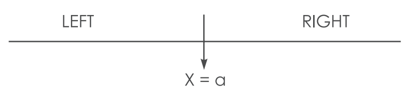
Whenever $0 < |x - a| < \delta$

From the above definition it is interesting to note $\lim_{x \to a} f(x)$ that may exist, even if the function f(x) is not defined at x = a. Sometimes both the things may happen, i.e.,

(i) The function is defined at x = a, and

(ii) $\lim_{x \to a} f(x)$ also exists.

Right hand and left hand limits



The variable point x can approach '*a*' either from the left or from the right. These respective approaches are indicated by writing $x \to a^-$ and $x \to a^+$

When

$\lim_{x \to a^-} f(x) = l_1$ and $\lim_{x \to a^+} f(x) = l_2$ then we can say $\lim_{x \to a} f(x) = l$ if and only if $l_1 = l_2 = l$

Methods of finding limit of a function f(x) as x tends to a finite quantity say 'a'

There are three methods for finding limit of a function f(x) as x tends to a finite quantity say 'a':

(i) Method of factors

(ii) Method of substitution

(iii) Method of rationalization.

In method I, if f(x) is of the form $\frac{g(x)}{h(x)}$ factorise g(x) and h(x), cancel the common factors and then put the value of x.

Illustration 24. *Find the value of* $\underset{x \to 1}{Lt} \frac{x^3 - 1}{x^2 - 1}$.

Solution:

Now, $\frac{x^3-1}{x^2-1}=\frac{(x-1)(x^2+x+1)}{(x-1)(x+1)}$

$=\frac{x^2+x+1}{x+1}$ [As $x \to 1, x \neq 1$ i.e $x-1 \neq 0$]

$\therefore \ \underset{x\to 1}{\text{Lt}} \frac{x^3-1}{x^2-1} = \underset{x\to 1}{\text{Lt}} \frac{x^2+x+1}{x+1}$

$=\frac{1+1+1}{1+1}=\frac{3}{2}$.

Method II

The following steps are involved:

(i) Put x = a + h where h is very small but ≠ 0, i.e. x → a, h→0.

(ii) Simplify numerator and denominator and cancel common powers of h.

(iii) Put h = 0.

The result is the required limit.

Illustration 25. Evaluate $\underset{x\to a}{\text{Lt}} \frac{x^n-a^n}{x-a}$.

Solution:

Put x = a + h where h is very small, then

$$\frac{x^n-a^n}{x-a}=\frac{(a+h)^n-a^n}{h}$$

$$=\frac{a^n\left[\left(1+\frac{h}{a}\right)^n-1\right]}{h}$$

$$=\frac{a^n\left[\left(1+\frac{nh}{a}+n(n-1)\frac{h^2}{a^2}+\cdots\right)-1\right]}{h}$$

$$=\frac{a^n\left[n\frac{h}{a}+n(n-1)\frac{h^2}{a^2}+\cdots\right]}{h}$$

$$=a^n\left[\frac{n}{a}+n(n-1)\frac{h}{a^2}+\cdots\right]$$

$$\therefore \ \underset{x\to a}{\text{Lt}} \frac{x^n-a^n}{x-a}=\underset{h\to 0}{\text{Lt}}\, a^n\left[\frac{n}{a}+n(n-1)\frac{h}{a^2}+\cdots\right]$$

$$=a^n\cdot\frac{n}{a}=na^{n-1}$$

Method III. (Rationalisation)

This method is useful where radical signs are involved either in the numerator or denominator. The numerator or denominator (as required) is rationalized and limit taken. The following **Illustration** will make the method clear.

Illustration 26. Evaluate $\underset{x\to 0}{\text{Lt}} \dfrac{\sqrt{1+x}-\sqrt{1-x}}{x}$

Solution:

Rationalising the numerator, we get

$$\frac{\sqrt{1+x}-\sqrt{1-x}}{x} = \frac{\sqrt{1+x}-\sqrt{1-x}}{x} \times \frac{\sqrt{1+x}+\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}$$

$$= \frac{2x}{x\left[\sqrt{1+x}+\sqrt{1-x}\right]}$$

$$= \frac{2}{\sqrt{1+x}+\sqrt{1-x}} \qquad [\text{As } x \to 0, x \neq 0]$$

$$\therefore \underset{x\to 0}{\text{Lt}} \frac{\sqrt{1+x}-\sqrt{1-x}}{x} = \underset{x\to 0}{\text{Lt}} \frac{2}{\sqrt{1+x}-\sqrt{1-x}} = 1$$

Infinite Limits

For finding the limit of $f(x) = \dfrac{g(x)}{h(x)}$ as a $\to \propto$, we divide the numerator and denominator by highest power of x occurring in f(x) (numerator or denominator whichever has higher power of x) and then use $\dfrac{1}{x}, \dfrac{1}{x^2}$, etc.$\to$ 0 as x $\to \propto$.

Illustration 27. Evaluate

$$\underset{x\to\infty}{\text{Lt}} \frac{(x+1)(2x+3)}{(x+2)(3x+4)}$$

Solution:

Now $\dfrac{(x+1)(2x+3)}{(x+2)(3x+4)} = \dfrac{2x^2+5x+3}{3x^2+10x+8}$

$$= \frac{2+\frac{5}{x}+\frac{3}{x^2}}{3+\frac{10}{x}+\frac{8}{x^2}} \qquad [\text{Dividing by } x^2]$$

$$\therefore \underset{x\to\infty}{\text{Lt}} \frac{(x+1)(2x+3)}{(x+2)(3x+4)} = \frac{2+5\times 0+3\times 0}{3+10\times 0+8\times 0} \quad \left[\text{As } x \to \infty, \frac{1}{x} \text{ as well as } \frac{1}{x^2} \to 0\right]$$

$$= \frac{2}{3}$$

Alternative approach:

$$\underset{x\to\infty}{\text{Lt}} \frac{(x+1)(2x+3)}{(x+2)(3x+4)}$$

$$= \underset{x\to\infty}{\text{Lt}} \frac{\left(1+\frac{1}{x}\right)\left(2+\frac{3}{x}\right)}{\left(1+\frac{2}{x}\right)\left(3+\frac{4}{x}\right)} = \frac{1\cdot 2}{1\cdot 3} = \frac{2}{3}$$

Illustration 28. $Lt_{x \to a} \dfrac{x\sqrt{x} - a\sqrt{a}}{x - a}$.

Solution:

$$Lt_{x \to a} \frac{x^{\frac{3}{2}} - a^{\frac{3}{2}}}{x - a} = \frac{3}{2} a^{\frac{1}{2}}.$$

Illustration 29. If $Lt_{(x \to 2)} \dfrac{x^n - 2^n}{x - 2} = 80$, find 'n'

Solution:

$n2^{n-1} = 80$

Or $n2^n = 160 = 5\ 2^5$

Or $n = 5$

Illustration 30. $Lt_{(x \to 0)} \dfrac{(1+x)^6-1}{(1+x)^2-1}$.

Solution:

$$\frac{Lt_{(x \to 0)} \dfrac{(1+x)^6-1}{x}}{Lt_{(x \to 0)} \dfrac{(1+x)^2-1}{x}} = \frac{6}{2} = 3.$$

SOLVED ILLUSTRATIONS

Illustration 31. $\lim_{x \to 2} \dfrac{x^2 - 4}{x - 2} = 4.$

For if x = 2 + h, whether h be positive or negative,

$$\frac{x^2-4}{x-2} = \frac{(x-2)(x+2)}{(x-2)} = \frac{h(4+h)}{h} = 4+h$$

and by taking h numerically small, the difference of $\dfrac{x^2-4}{x-2}$ and 4 can be made as small as we like. It may be noted here that however small h may be, as $h \neq 0$, one can cancel the factor $(x - 2)$ i.e., h between numerator and denominator here. Hence $\lim_{x \to 2} \dfrac{x^2-4}{x-4} = 4.$ But for x = 2, the function $\dfrac{x^2-4}{x-2}$ is undefined as we cannot cancel the factor x – 2, which is equal to zero.

Now writing $f(x) = \dfrac{x^2-4}{x-2}, \lim_{x \to 2} f(x) = 4$ whereas f(2) does not exist.

Illustration 32. For f (x) = | x |, find, $\lim_{x \to 0} f(x)$.

Solution:

$$f(x) = |x| \text{ means } f(x) = \begin{cases} x, & \text{for } x \geq 0 \\ -x, & \text{for } x < 0 \end{cases}$$

$\lim_{x\to 0+} f(x) = \lim_{x\to 0+} |x| = \lim_{x\to 0} x = 0$ [x → 0^+ means x is approaching 0 from right i.e x > 0 now. Also |x| = x for x>0]

$\lim_{x\to 0-} f(x) = \lim_{x\to 0-} |x| = \lim_{x\to 0}(-x) = 0.$ [x → 0^- means x is approaching 0 from left i.e x < 0 now. Also |x| = – x for x< 0]

So $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^-} f(x) = 0$

∴ $\lim_{x\to 0} f(x) = 0.$

Illustration 33. Find analytically $\lim_{x\to 3}\sqrt{x-3},$ if it exists.

Solution:

$\lim_{x\to 3+}\sqrt{x-3} = 0,$ but $\lim_{x\to 3-}\sqrt{x-3}$ does not exist. Because when x → 3^- then x is approaching 3 from left which means x is definitely less than 3. Thus (x – 3) is negative and square root of negative quantity is never real but imaginary.

∴ the limit does not exist.

Alternative approach:

Put x = 3 + h, as x → 3, h → 0

$$\lim_{x\to 3}\sqrt{x-3} = \lim_{h\to 0}\sqrt{3+h-3} = \lim_{h\to 0}\sqrt{h}$$

$\lim_{h\to 0+}\sqrt{h} = 0$, but $= \lim_{h\to 0-}\sqrt{h}$ does not exist (as the h near to but less than zero corresponds no real value of $\sqrt{h}$).

∴ $\lim_{x\to 3}\sqrt{x-3}$ does not exist.

Note : At x = 3, f (x) = $\sqrt{x-3} = \sqrt{3-3} = 0$ ∴ f(3) exists.

Illustration 34. Do the following limits exist? If so find the values

(i) $\lim_{x\to -2}\frac{1}{x+2}$ (ii) $\lim_{x\to 0}\frac{1}{x}$ (iii) $\lim_{x\to 1}\left\{\left(x^2-1\right)+\frac{(x-1)^2}{x-1}\right\}$

Solution:

(i) $\lim_{x\to -2}\frac{1}{x+2} = \lim_{h\to 0}\frac{1}{-2+h+2} = \lim_{h\to 0}\frac{1}{h}$

Now $\lim_{x\to 0+}\frac{1}{h} = +ve;\ \lim_{h\to 0-}\frac{1}{h} = -ve.$ As the two limits are not same, so the limit does not exist.

(ii) $\lim_{x\to 0+}\frac{1}{x} = +\infty,\ \lim_{h\to 0-}\frac{1}{x} = -\infty.$ The limit doesn't exist as the two values are unequal.

(iii) $\lim_{x\to1}\left\{\left(x^2-1\right)+\frac{(x-1)(x-1)}{(x-1)}\right\}=\lim_{x\to1}\left\{\left(x^2-1\right)+(x-1)\right\}$, [As $x \to 1, x \neq 1$ i.e $x-1 \neq 0$]

$=\lim_{x\to1}\left(x^2+x-2\right)=(1+1-2)=0.$

On putting the limiting value of x, the value of the function exists and its value is 0.

Distinction between $\lim_{x\to a} f(x)$ and $f(a)$

By $\lim_{x\to a} f(x)$ we mean the value of f(x) when x has any arbitary value very near 'a' but not 'a'. The quantity f (a) is the value of f(x), when x is exactly equal to 'a'.

Note : The following cases may arise :

(i) $f(a)$ does not exist, but $\lim_{x\to a} f(x)$ exists.

Such situation is explained in example 28 above.

(ii) $f(a)$ exists, but $\lim_{x\to a} f(x)$ does not exist.

Such situation is explained in example 30 above.

(iii) $f(a)$ and both exist, $\lim_{x\to a} f(x)$ both exist but unequal.

Let $f(x)$ = 0, for $x \neq 0$

= 1, for $x = 0$

$\lim_{x\to0+} f(x)=0=\lim_{x\to0-} f(x)$ and f(0) = 1, unequal values.

(iv) $f(a)$ and $\lim_{x\to a} f(x)$ both exist and equal.

Such situation is explained in example 29 above.

(v) neither $f(a)$ nor, $\lim_{x\to a} f(x)$ exists.

Fundamental Theorem on Limits:

If $\lim_{x\to a} f(x)=l$ and $\lim_{x\to a}\phi(x)=m$, where l and m are finite quantities then

1. $\lim_{x\to a}\left[f(x)\pm\phi(x)\right]=1\pm m$

2. $\lim_{x\to a}\left[f(x).\ \phi(x)\right]=$ lm.

3. $\lim_{x\to a}\frac{f(x)}{\phi(x)}=\frac{l}{m}, m\neq0$

4. If $\lim_{x\to a}\phi(x)=b$ and $\lim_{x\to b} f(u)=f(b)$ then

$$\lim_{x\to b} f\{\phi(x)\} = f\left\{\lim_{x\to b}\phi(x)\right\} = f(b).$$

5. $\lim_{x\to a}$ k = k, where k = Constant

6. $\lim_{x\to a} k.f(x) = k\lim_{x\to a} f(x)$, where k = Constant

Illustration 35. Evaluate, $\lim_{x\to 1}\frac{x^2+3x-1}{2x+4}$.

Solution:

As the limit of the denominator ≠ 0, we get

$$\text{Required limit } = \frac{\lim_{x\to 1}\left(x^2+3x-1\right)}{\lim_{x\to 1}(2x+4)} = \frac{\lim_{x\to 1}x^2+\lim_{x\to 1}3x-\lim_{x\to 1}1}{\lim_{x\to 1}2x+\lim_{x\to 1}4} \text{ (by theorem 1)}$$

$$= \frac{1+3.1-1}{2.1+4} = \frac{3}{6} = \frac{1}{2}$$

We have not applied the definition to save labour. If we substitute x = 1, we get the value of the function $=\frac{1}{2}$ (equal to the limit the value as x →1). Practically this may not happen always, as shown below.

Illustration 36. Evaluate $\lim_{x\to 1}\frac{x^2-3x+2}{x^2-4x+3}$.

$$\text{Required limit} = \lim_{x\to 1}\frac{(x-1)(x-2)}{(x-1)(x-3)} = \lim_{x\to 1}\frac{x-2}{x-3} \;[\text{As } x\to 1, x\neq 1 \text{ i.e } x-1\neq 0]$$

$$= \frac{1-2}{1-3} = \frac{1}{2}.$$

$$\lim_{x\to 1}\frac{(x-1)(x-2)}{(x-1)(x-3)} = \lim_{x\to 1}\frac{x-2}{x-3} = \frac{1-2}{1-3} = \frac{1}{2}.$$

If we put x = 1, the function becomes $\frac{0}{0}$ which is undefined. Further limit of the denominator is zero, we cannot apply theorem 2, hence cancelling the common factor (x – 1) which is ≠ 0 as x →1, we obtain the above result.

Illustrations related to variable tends of infinity:

Illustration 37. Find the limit of $\lim_{x\to\infty}\left(2x^2-5x+2\right)$

Solution:

Now $2x^2 - 5x + 2 = x^2\left(2-\frac{5}{x}+\frac{2}{x^2}\right)$

$$\therefore \lim_{x\to\infty}\left(2x^2-5x+2\right)=\lim_{x\to\infty}x^2\left(2-\frac{5}{x}+\frac{2}{x^2}\right)=\lim_{x\to\infty}x^2\times\lim_{x\to\infty}\left(2-\frac{5}{x}+\frac{2}{x^2}\right)$$

Now $\lim_{x\to\infty}x^2=\infty$; $\lim_{x\to\infty}2=2$; $\lim_{x\to\infty}\frac{5}{x}=0$; $\lim_{x\to\infty}\frac{2}{x^2}=0$ [As $x\to\propto$, $\frac{1}{x}$ & $\frac{1}{x^2}=0$]

$$\therefore \lim_{x\to\infty}\left(2x^2-5x+2\right)=\infty\times 2=\infty.$$

Illustration 38. Find $\lim_{x\to\infty}\frac{4x^5+2x^3-5}{7x^8+x^4+2}$

Solution:

Of all the terms in numerator and denominator the highest power of x is 8. We now divide both the numerator and the denominator by x^8 to avoid the undefined form $\frac{\infty}{\infty}$. So we get

Given limit = $\lim_{x\to\infty}\dfrac{\frac{4}{x^3}+\frac{2}{x^5}-\frac{5}{x^8}}{7+\frac{1}{x^4}+\frac{2}{x^8}}=\lim_{u\to 0}\dfrac{4u^3+2u^5-5u^8}{7+u^4+2u^8}$ (put $\frac{1}{x}=u$, so as x → ∞ , $u\to 0$)

$$=\frac{\lim_{u\to 0}\left(4u^3+2u^5-5u^8\right)}{\lim_{u\to 0}(7+u^4+2u^8)}$$

$$=\frac{4\lim_{u\to 0}u^3+2\lim_{u\to 0}u^5-5\lim_{u\to 0}u^8}{7+\lim_{u\to 0}u^4+2\lim_{u\to 0}u^8}=\frac{0+0-0}{7+0+0}=\frac{0}{7}=0$$

Illustration 39. Find $\lim_{x\to\infty}\frac{5-2x^2}{3x+5x^2}$.

Solution:

Given limit $=\lim_{x\to\infty}\dfrac{\frac{5}{x^2}-2}{\frac{3}{x}+5}=\lim_{u\to 0}\dfrac{5u^2-2}{3u+5}$ $\left(Let\ \frac{1}{x}=u,\ as\ x\to\infty,\ u\to 0\right)$

$$=\frac{5\lim_{u\to 0}u^2-2}{3\lim_{u\to 0}u+5}=\frac{0-2}{0+5}=-\frac{2}{5}.$$

Illustration related to rationalisation:

Illustration 40. Find the value of : $\lim_{h\to 0}\frac{\sqrt{x+h}-\sqrt{x}}{h}$

$$\lim_{h\to 0}\frac{\sqrt{x+h}-\sqrt{x}}{h}=\lim_{h\to 0}\frac{\left(\sqrt{x+h}-\sqrt{x}\right)\left(\sqrt{x+h}+\sqrt{x}\right)}{h\left(\sqrt{x+h}+\sqrt{x}\right)}=\lim_{h\to 0}\frac{x+h-x}{h\left(\sqrt{x+h}+\sqrt{x}\right)}$$

$$= \lim_{h\to 0}\frac{h}{h\left(\sqrt{x+h}+\sqrt{x}\right)} = \lim_{h\to 0}\frac{1}{\sqrt{x+h}+\sqrt{x}}, \text{[As } h \to 0, h \neq 0) \quad = \frac{1}{\sqrt{x+0}+\sqrt{x}} = \frac{1}{2\sqrt{x}}.$$

Illustrations related to left hand and right hand limits:

Illustration 41. A function is defined as

$f(x)$	$= x^2$,	for	$x > 1$
	$= 4.1$,	for	$x = 1$
	$= 2x$,	for	$x < 1$.

Does $\lim_{x\to 1} f(x)$ exist?

Solution:

We have to find R.H. limit and L.H. limit i.e $\lim_{x\to 1+} f(x)$ and $\lim_{x\to 1-} f(x)$.

Now $\lim_{x\to 1+} f(x) = \lim_{x\to 1+} x^2 = 1^2 = 1$, [As $f(x) = x^2$ for $x > 1$]

Again $\lim_{x\to 1+} f(x) = \lim_{x\to 1+} 2x = 2\times 1 = 2$, [As $f(x) = 2x$ for $x < 1$]

Since the values of R.H. L and L.H.L are not equal, so $\lim_{x\to 1} f(x)$ doesn't exist.

Illustration 42.

Evaluate (i) $\lim_{x\to 0}\frac{4x+|x|}{3x+|x|}$ (ii) $\lim_{x\to 2}\frac{|x-2|}{x-2}$

Solution:

(i) R.H. Limit $= \lim_{x\to 0+}\frac{4x+x}{3x+x} = \lim_{x\to 0}\frac{5x}{4x} = \frac{5}{4}$, [As $x \to 0$, So $x \neq 0$ and also for $x > 0$, $|x| = x$]

L.H. Limit $= \lim_{x\to 0-}\frac{4x-x}{3x-x} = \lim_{x\to 0-}\frac{3x}{2x} = \frac{3}{2}$ [As $x \to 0$, So $x \neq 0$, and for $x < 0$, $|x| = -x$]

$\therefore$ the given limit doesn't exist, as the values of R.H.L and L.H.L are unequal.

(ii) R.H.L $= \lim_{x\to 2+}\frac{|x-2|}{x-2} = \lim_{x\to 2}\frac{x-2}{(x-2)} = 1$. [As $x \to 2$, $x \neq 2$ i.e $(x-2) \neq 0$. Also for $|x-2| > 0$ i.e., for $x > 2$,

$|x-2| = x-2$

L.H.L $= \lim_{x\to 2-}\frac{|x-2|}{x-2} = \lim_{x\to 2-}\frac{-(x-2)}{(x-2)} = -1$ for $|x-2| < 0$ i.e., for $x < 2$, $|x-2| = -(x-2)$

As the values of R.H.L and L.H.L are unequal, so the given limit does not exist.

Some Useful Limits :

(A) $\lim_{x\to 1}(1+x)^{1/x} = e$

(B) $\lim_{x\to 0}\frac{1}{x}\log_e(1+x) = 1$

(C) $\lim_{x\to 0}\frac{e^x-1}{x} = 1$

(D) $\lim_{x\to 0}\frac{a^x-1}{x} = \log_e a$

(E) $\lim_{x\to a}\frac{x^n-a^n}{x-a} = na^{n-1}$

(F) $\lim_{x\to 0}\frac{(1+x)^n-1}{x} = n.$

Illustration 43. Evaluate $\lim_{x\to 0}\frac{e^{3x}-1}{x}$

Solution:

Put $3x = u$, As $x \to 0$ $u \to 0$.

$$\lim_{x\to 0}\frac{e^{3x}-1}{x} = \lim_{u\to 0}\frac{e^u-1}{u/3} = 3\times\lim_{u\to 0}\frac{e^u-1}{u}. = 3\times 1\,(by\ C) = 3.$$

Illustration 44. Evaluate $\lim_{x\to 0}\frac{\log(1+6x)}{x}$

Solution:

Put $6x$ = u, As $x \to 0$, $u \to 0$

Given limit $= \lim_{u\to 0}\frac{\log(1+u)}{u/6} = 6\times\lim_{u\to 0}\frac{\log(1+u)}{u}$ = 6 × 1 (by B) = 6.

Illustration 45. $\lim_{x\to 0}\frac{a^x-b^x}{x}$

Solution:

Given limit $= \lim_{x\to 0}\frac{a^x-1-b^x+1}{x} = \lim_{x\to 0}\frac{a^x-1}{x} - \lim_{x\to 0}\frac{b^x-1}{x}$

= log a – log b [by (D) above] = log $\frac{a}{b}$.

Illustration 46. $\lim_{x\to 0}\frac{e^{ax}-e^{bx}}{x}$

Solution:

Given limit $= \lim_{x\to 0}\frac{(e^{ax}-1)-(e^{bx}-1)}{x} = \lim_{x\to 0}\frac{e^{ax}-1}{x} - \lim_{x\to 0}\frac{e^{bx}-1}{x}$

$= \lim_{u\to 0}\frac{e^u-1}{u/a} - \lim_{v\to 0}\frac{e^v-1}{v/b}$ [Let ax = u, As $x\to 0$ $u\to 0$ also. Similarly for $bx = v$, As $x\to 0$, $v\to 0$ also]

$= a\lim_{u\to 0}\frac{e^u-1}{u} - b\lim_{v\to 0}\frac{e^v-1}{v}$

= a. 1 – b.1 = a – b.

CONTINUITY

Introduction :

A function is said to be continuous if its graph is a continuous curve without any break. If, however, there is any break in the graph, then function is not continuous at that point.

If for a value of k, the limit of f(x) does not exist i.e., if on the curve of f(x) a point is absent, the graph will be discontinuous i.e., not continuous.

A function f(x) is said to be continuous at $x = a$, when $\lim_{x\to a} f(x)$ if exists is finite and is equal to f(a)

i.e., $\lim_{x\to a+} f(x) = \lim_{x\to a-} f(x) = f(a)$

i.e., f (a + 0) = f (a – 0) = f (a) briefly or $\lim_{x\to 0} f(a+h) = f(a)$

Thus we are to find following three values :

(i) $\lim_{x\to a+} f(x)$, (ii) $\lim_{x\to a-} f(x)$ (iii) f(a)

If however all these values are equal, then $f(x)$ is continuous at $x = a$, otherwise it is discontinuous.

Illustration 47. Show that $f(x) = 3x^2 - x + 2$ is continuous at $x = 1$.

Solution:

Now, $\lim_{x\to 1+} f(x) = \lim_{x\to 1+}(3x^2 - x + 2) = \lim_{h\to 0}\left\{3(1+h)^2 - (1+h) + 2\right\}$ [Putting x = 1 + h as x → 1, h → 0]

$= \lim_{h\to 0}(3h^2 + 5h + 4) = 3.0 + 5.0 + 4 = 4$

and $\lim_{h\to 1-} f(x) = \lim_{x\to 1-}(3x^2 - x + 2) = \lim_{h\to 0}\left\{3(1-h)^2 - (1-h) + 2\right\}$ [Putting $x = 1 - h$, $h \to 0$ as $x \to 1$]

$= \lim_{h\to 0}(3h^2 - 5h + 4) = 3.0 - 5.0 + 4 = 4$

Again $f(1) = 3.1^2 - 1 + 2 = 4$

Thus we find that all the values are equal.

∴ $f(x)$ is continuous at $x = 1$;

ε - δ Definition :

Again corresponding to definition of limit, we may define the continuity of a function as follows :

The functions $f(x)$ is continuous at $x = a$, if $f(a)$ exists and for any pre-assigned positive quantity ε, however small we can determine a positive quantity δ, such that $|f(x) - f(a)| < \varepsilon$, for all values of x satisfying $|x - a| < \delta$.

Some Properties :

1. The sum or difference of two continuous functions is a continuous function
   i.e., $\lim_{x\to a}\left\{f(x) \pm \phi(x)\right\} = \lim_{x\to a} f(x) \pm \lim_{x\to a} \phi(x)$.
2. Product of two continuous functions is a continuous function.
3. Ratio of two continuous functions is a continuous function, provided the denominator is not zero.

Continuity in an Interval, at the End Points:

A function is said to be continuous over the interval (open or closed) including the end points if it is continuous at every point of the same interval.

Let *c* be any point in the interval (*a*, *b*) and if $\lim_{x \to c+} f(x) = \lim_{x \to c-} f(x) = f(c)$, then f(x) is continuous in the interval (*a*, *b*)

A function *f*(*x*) is said to be continuous at the *left end* '*a*' of an interval $a \leq x \leq b$ if $\lim_{x \to 0} f(a+h) = f(a)$, and at the *right end* '*b*' if $\lim_{x \to b-} f(x) = f(b)$.

Discontinuity at a Point: If at any point *x* = *a* in its domain, at least one of values and *f*(*a*) be different from the others, then *f*(*a*) fails to be continuous at that point, i.e., at *x* = *a*.

Illustration 48. Discuss the continuity of *f* (*x*) at *x* = 4, where

$$f(x) = 2x + 1, x \neq 4$$
$$= 8, x = 4.$$

Solution:

Here $\lim_{x \to 4-} f(x) = \lim_{x \to 4} (2x+1) = 2.4+1 = 9$ [$x \longrightarrow 4^+$ means *x* is approachig 4 from right i.e $x > 4$. Also $f(x) = 2x + 1$ for $x \neq 4$]

$\lim_{x \to 4-} f(x) = \lim_{x \to 4} (2x+1) = 2.4+1 = 9$ [$x \longrightarrow 4^-$ means *x* is approachig 4 from left i.e $x < 4$. Also $f(x) = 2x + 1$ for $x \neq 4$]

$f(4) = 8$, which is different from the previous two values.

$\therefore$ $f(x)$ is not continuous at $x = 4$.

SOLVED ILLUSTRATIONS

Illustration 49. Show that $f(x) = 2x + 3$ is continuous at $x = 1$.

Solution:

$\lim_{x \to 1+} f(x) = \lim_{x \to 1} (2x+3) = 2.1+3 = 5.$

$\lim_{x \to 1-} f(x) = \lim_{x \to 1} (2x+3) = 2.1+3 = 5.$ and at x = 1, $f(1) = 2.1 + 3 = 5$

$\therefore \lim_{x \to 1+} f(x) = \lim_{x \to 1-} f(x) = f(1) = 5$; $\therefore f(x)$ is continuous at $x = 1$.

Illustration 50. Discuss the continuity of $f(x) = |x|$ at x = 0.

Solution:

$$f(x) = |x| \text{ means } f(x) = x \text{ for } x > 0$$
$$= 0 \text{ for } x = 0$$
$$= -x \text{ for } x < 0.$$

$\lim\limits_{x \to 0+} f(x) = \lim\limits_{x \to 0} x = 0$ [$x \to 0^+$ means x is approaching 0 from right i.e $x > 0$ now. Also $f(x) = x$ for $x > 0$]

$\lim\limits_{x \to 0-} f(x) = \lim\limits_{x \to 0} (-x) = 0$ [$x \to 0^-$ means x is approaching 0 from left i.e $x < 0$ now. Also $f(x) = -x$ for $x < 0$]

$f(0) = f(x)$ at $x = 0 = 0$

$\therefore \lim\limits_{x \to 0+} f(x) = \lim\limits_{x \to 0-} f(x) = f(0) = 0$; $\therefore f(x)$ is continous at $x = 0$.

Illustration 51. Show that $f(x) = \dfrac{1}{x-1}$ is discontinuous at $x = 1$.

Solution:

$f(1) = \dfrac{1}{1-1} = \dfrac{1}{0}$, undefined.

Now $\lim\limits_{x \to 1+} f(x) = \lim\limits_{h \to 0} \dfrac{1}{1+h-1} = \lim\limits_{h \to 0} \dfrac{1}{h} = +\infty$ (Putting $x = 1 + h$, As $x \to 1$, $h \to 0$)

And $\lim\limits_{x \to 1-} f(x) = \lim\limits_{h \to 0} \dfrac{1}{1-h-1} = \lim\limits_{h \to 0} \dfrac{1}{-h} = -\infty$ (Putting $x = 1 - h$, As $x \to 1$, $h \to 0$)

We find right hand limit is not equal to left hand limit.

$\therefore$ at $x = 1$, f (x) is discontinuous.

Illustration 52. Discuss the continuity at $x = 2$ where

$$f(x) = \begin{cases} 4x + 8, & x \neq 2 \\ 12, & x = 2 \end{cases}$$

Solution:

Now $\lim\limits_{x \to 2+} f(x) = \lim\limits_{h \to 0} 4(2+h) + 8 = 16$ (Putting $x = 2 + h$ As $x \to 2$, $h \to 0$)

And $\lim\limits_{x \to 2-} f(x) = \lim\limits_{h \to 0} 4(2-h) + 8 = 16.$ [Putting $x = 2 - h$, $h \to 0$, As $x \to 2$]

Again at $x = 2$, $f(x) = f(2) = 12$.

Three values are not equal. Hence f(x) is discontinuous at $x = 2$.

OBJECTIVE QUESTIONS :

1. If $f(x) = (x - 1)(x - 2)(x - 3)$, find f(0) [Ans. – 6]
2. If $f(x) = x + |x|$, find f(– 2) [Ans. 0]
3. If $f(x) = \dfrac{x^2 - 6x + 8}{x^2 + 8x + 12}$ find f(0) [Ans. $2/3$]
4. Given $f(x) = x$, $F(x) = \dfrac{x^2}{x}$, is $F(x) = f(x)$ always? [Ans. equal for $x \neq 0$]

5. If f(x) = 2 + x, x < 3 [Ans. 4]
   = 7 – x, x ≥ 3, find f(3)
6. If f(x) = (x – 2) (x – 3) (x + 4) find f(3) [Ans. 0]
7. Given $f(x)=\sqrt{x}$, for what value of x, f(x) is unreal? [Ans. Any negative value]
8. Find the range of the function $f(x)=\frac{x-2}{2-x}, x\neq 2$ [Ans. – 1]
9. If f(x) = x + 2x^3, find – f(– x) [Ans. x + 2x^3]
10. $f(x)=\frac{|x|}{x}, x\neq 0$ and is a real number, find $|f(c)-f(-c)|$. [Ans. 2]
11. If f(x) = e^{3x+4}, find f(1) . f(2) . f(5) [Ans. e^{36}]
12. If $x=\frac{1}{3}$ find $f(x)=\frac{1-x}{1+x}$, [Ans. $\frac{1}{2}$]
13. If f(x) = x + | x | are f(2) and f(– 2) equal? [Ans. no]
14. Evaluate : (i) $\lim_{x\to 1}\frac{x^2+1}{x+1}$ (ii) $\lim_{x\to 1}\frac{x^2-1}{x-1}$ (iii) $\lim_{x\to 2}\frac{|x-2|}{x-2}$ [Ans. (i) 1, (ii) 2, (iii) does not exist]
15. Evaluate : (i) $\lim_{x\to 1}\frac{x^2+5x-6}{x-1}$ (ii) $\lim_{x\to 0}\frac{e^x-1}{x}$ [Ans. (i) 7 (ii) 1]
16. Evaluate : $\lim_{x\to -1}\frac{x^2+4x+3}{x^2-7x-8}$ [Ans. $\frac{-2}{9}$]
17. Find the value of $\lim_{x\to 0}\frac{1-\sqrt{1-x}}{x}$ [Ans. $\frac{1}{2}$]
18. If f(x) = x^2, evaluate : $\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$ [Ans. 2x]
19. Is the function f(x) = | x | continuous at x = 0 [Ans. Yes]
20. Evaluate : $\lim_{x\to\infty}\frac{6-5x^2}{4x+15x^2}$. [Ans. $-\frac{1}{3}$]
21. A functions is defined as follows :
   f(x) = 2x – 1, x < 3
   = k , x = 3
   = 8 – x, x > 3.
   For what value of k, f(x) is continuous at x = 3? [Ans. 5]
22. A function is defined as follows :
   f(x) = 1, x > 0
   = 0, x = 0
   = – 1 x < 0. Is the function is continuous at x = 0 [Ans. no]

DIFFERENTIATION

Introduction :

The idea of limit as discussed previously will now be extended at present in differentiating a function f(x) with respect to x (the independent variable). For this let us know at first what the term 'increment' means.

Increment : By increment of a variable we mean the difference of initial value from the final value.

i.e., Increment = final value – initial value.

Let x change its value from 1 to 4, increment of x = 4 – 1 = 3.

Again if x changes from 1 to – 2, increment = – 2 – 1 = – 3.

(i.e., increment may be positive or negative).

Symbols : Increment of x will be denoted by h or, δx (delta x) or Δx (delta x) and that of y will be represented by k or, δy or, Δy.

If in y = f(x), the independent variable x changes to x + δx, then increment of x = x + δx – x = δx (≠0).

So y = f(x) changes to y = f(x + δx).

∴ increment of y = f(x + δx) – f(x) [as y = f(x)]

Now the increment ratio $\frac{\delta y}{\delta x} = \frac{f(x+\delta x)-f(x)}{\delta x} = \frac{f(x+h)-f(x)}{h}$ [assuming δx = h]

If the ratio $\frac{\delta y}{\delta x}$ tends to a limit, as δx→0 from either side, then this limit is known as the differential coefficient or derivative of y [= f(x)] with respect to x. The operation of finding differential coefficient (or derivetive) is called Differentiation.

Illustration 53. If y = $2x^2$, then y + dy = 2 $(x + dx)^2$, δy = 2 $(x + dx)^2 – 2x^2$

$$\therefore \frac{\delta y}{\delta x} = \frac{2(x+\delta x)^2 - 2x^2}{\delta x}$$

Again for $y = \frac{1}{x^5}$, $\delta y = \frac{1}{(x+\delta x)^5} - \frac{1}{x^5}$

Definition : A function y = f(x) is said to be derivable at x if $\lim_{\delta x \to 0} \frac{\delta y}{\delta x}$ or, $\lim_{\delta x \to 0} \frac{f(x+\delta x)-f(x)}{\delta x}$ or, $\lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$ exists and equal to *l*. Now this limit *l* that exists is known as derivative (or differential co-efficient) of y [or f(x)] with respect to x.

Symbols : Derivative of y [= f(x)] w.r.t.x (with respect to x) is denoted by

$\frac{dy}{dx}$ or, f ′ (x), or, $\frac{d}{dx}[f(x)]$ or, Dy or, D [f(x)] or, y_1

Now $\frac{dy}{dx} = \lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \lim_{\delta x \to 0} \frac{f(x+\delta x)-f(x)}{\delta x} = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$, provided this limit exists.

Note : $\frac{dy}{dx}$ does not mean the product of $\frac{d}{dx}$ with y. The notation $\frac{d}{dx}$ stands as a symbol to denote the operation of differentiation only. Read $\frac{dy}{dx}$ as 'dee y by dee x'.

SUMMARY :

The whole process for calculating f' (x) or $\frac{dy}{dx}$ may be summed up in the following stages :

1. Let the independent variable x has an increment h and then find the new value of the function f (x + h).
2. Find f(x + h) – f(x).
3. Divide the above value by h i.e., find $\frac{f(x+h)-f(x)}{h}$.
4. Calculate $\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = f'(x)$

SOME USEFUL DERIVATIVES :

1. $\frac{d}{dx}x^n = nx^{n-1}$
2. $\frac{d}{dx}.\frac{1}{x^n} = -\frac{n}{x^{n+1}}$.
3. $\frac{d}{dx}e^x = e^x$. [e = constant = 2.718 (Approx)]
4. $\frac{d}{dx}a^x = a^x \log_e a$.
5. $\frac{d}{dx}\log_e x = \frac{1}{x}$.
6. $\frac{d}{dx}(\log_a x) = \frac{1}{x}\log_a e$. [a = Constant other than 'e']
7. $\frac{dc}{dx} = 0$ (c = constant)
8. $\frac{d}{dx}(u \pm v) = \frac{du}{dx} \pm \frac{dv}{dx}$ [u and v are functions of x]
9. $\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$ [u and v are functions of x]

This formula is knon as Product formula because the function to be differentiated is expressed as product of two functions

10. $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$ [u and v are functions of x]

This formula is known as Division formula because the function to be differentiated is expressed as quotient of two functions.

11. $\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx}$

Illustration 54. $\frac{d}{dx}\left(x^4\right) = 4x^3$; $\frac{d}{dx}(x) = 1.x^{1-1} = 1x^0 = 1.1 = 1.$

$$\frac{d}{dx}\left(\frac{1}{x}\right) = \frac{d}{dx}\left(x^{-1}\right) = -1.x^{-1-1} = -1x^{-2} = \frac{-1}{x^2}$$

$$\frac{d}{dx}.\left(\sqrt{x}\right) = \frac{d}{dx}\left(x^{1/2}\right) = \frac{1}{2}.x^{-1/2} = \frac{1}{2\sqrt{x}};$$

$$\frac{d}{dx}.\left(\frac{1}{\sqrt{x}}\right) = \frac{d}{dx}.\left(x^{-1/2}\right) = -\frac{1}{2}.x^{-3/2} = \frac{-1}{2x^{3/2}}; \frac{d}{dx}\left(x\sqrt{x}\right) = \frac{d}{dx}.\left(x^{3/2}\right) = \frac{3}{2}x^{1/2} = \frac{3}{2}\sqrt{x};$$

Illustration 55. $\frac{d}{dx}(x^3 + x^2) = \frac{d}{dx}(x^3) + \frac{d}{dx}(x^2) = 3x^2 + 2x.$

Illustration 56. $\frac{d}{dx}(x^2.e^x) = x^2\frac{d}{dx}(e^x) + e^x\frac{d}{dx}x^2 = x^2e^x + e^x.2x = x^2e^x + 2xe^x.$

Illustration 57. $\frac{d}{dx}(2x^4) = 2\frac{d(x^4)}{dx} = 2.4x^3 = 8x^3.$

Illustration 58. If $y = \frac{x^2}{x+1}$ find $\frac{dy}{dx}$

Solution:

Let $y = \frac{u}{v}$ where $u = x^2$, $\frac{du}{dx} = \frac{d}{dx}\left(x^2\right) = 2x^{2-1} = 2x$

And $v = (x+1), \frac{dv}{dx} = \frac{d}{dx}(x+1) = \frac{d}{dx}x + \frac{d}{dx}.1 = 1 + 0 = 1$

Now $\frac{dy}{dx} = \frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u.\frac{dv}{dx}}{v^2} = \frac{(x+1).2x - x^2.1}{(x+1)^2} = \frac{2x^2 + 2x - x^2}{(x+1)^2} = \frac{x^2 + 2x}{(x+1)^2}$

Illustration 59. Find $\frac{dy}{dx}$ of the following functions :

(i) $x^4 + 4x$, (ii) $3x^5 - 5x^3 + 110$, (iii) $-2 + (4/5)x^5 - (7/8)x^8$.

Solution:

(i) Let $y = x^4 + 4x$.

$$\frac{dy}{dx} = \frac{d}{dx}\left(x^4 + 4x\right) = \frac{d}{dx}(x^4) + \frac{d}{dx}(4x) = 4.x^{4-1} + 4\frac{d}{dx}.x = 4x^3 + 4.$$

(ii) Let $y = 3x^5 - 5x^3 + 110$

$$\frac{dy}{dx} = \frac{d}{dx}\left(3x^5 - 5x^3 + 110\right) = \frac{d}{dx}\left(3x^5\right) - \frac{d}{dx}\left(5x^3\right) + \frac{d}{dx}.110$$

$= 3\frac{d}{dx}(x^5) - .5\frac{d}{dx}(x^3) + 0$ (as 110 is a constant number)

$= 3.5x^{5-1} - 5.3x^{3-1} = 15x^4 + 15x^2.$

(iii) Let $y = -2 + \frac{4}{5}x^5 - \frac{7}{8}x^8$

$$\frac{dy}{dx} = \frac{d}{dx}\left(-2 + \frac{4}{5}x^5 - \frac{7}{8}x^8\right) = \frac{d}{dx}(-2) + \frac{d}{dx}\left(\frac{4}{5}x^5\right) - \frac{d}{dx}\left(\frac{7}{8}x^8\right)$$

$= 0 + \frac{4}{5}\frac{d}{dx}(x^5) - \frac{7}{8}\frac{d}{dx}(x^8)$, (as – 2 is a constant number)

$= \frac{4}{5}.5x^{5-1} - \frac{7}{8}.8x^{8-1} = 4x^4 - 7x^7.$

Illustration 60. If $s = ut + \frac{1}{2}ft^2$, find $\frac{ds}{dt}$ when $t = 2$.

Solution:

$$\frac{ds}{dt} = \frac{d}{dt}\left(ut + \frac{1}{2}ft^2\right) = \frac{d}{dt}(ut) + \frac{d}{dt}\left(\frac{1}{2}ft^2\right) = u\frac{dt}{dt} + \frac{1}{2}f\frac{d}{dt}(t^2)$$

(here u, f, $\frac{1}{2}$ are constants & t is a variable, since we are to differentiate w.r.t. t)

$= u.1 + \frac{1}{2}f.2t^{2-1} = u + \frac{1}{2}.2ft = u + ft$

For $t = 2, \frac{ds}{dt} = u + 2f.$

ILLUSTRATIONS RELATED TO PRODUCT FORMULA:

Illustration 61. Differentiate (x + 1) $(2x^3 - 21)$ with respect to x.

Solution:

Let y = (x + 1) $(2x^3 - 21)$ = u. v where u = x + 1, $v = 2x^3 - 21$

$$\frac{du}{dx} = \frac{d}{dx}(x+1) = \frac{d(x)}{dx} + \frac{d}{dx}(1) = 1 + 0 = 1$$

$$\frac{dv}{dx} = \frac{d}{dx}(2x^3 - 21) = 2\frac{d}{dx}.(x^3) - \frac{d}{dx}(21) = 2.3x^{3-1} - 0 = 6x^2.$$

Now $\frac{dy}{dx} = \frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx} = (x+1).\ 6x^2 + (2x^3 - 21).1$

$= 6x^3 + 6x^2 + 2x^3 - 21 = 8x^3 + 6x^2 - 21.$

Illustration 62. $y = x(x^2 - 1)(x^3 + 2)$, find $\frac{dy}{dx}$.

Solution:

Let $y = uvw$, where $u = x$, $v = x^2 - 1$, $w = x^3 + 2$

$$\frac{du}{dx} = 1, \frac{dv}{dx} = 2x \text{ and } \frac{dw}{dx} = 3x^2$$

$$\frac{dy}{dx} = \frac{d}{dx}(uvw) = vw\frac{du}{dx} + uw\frac{dv}{dx} + uv\frac{dw}{dx}$$

$= (x^2 - 1)(x^3 + 2).1 + x(x^3 + 2).\ 2x + x(x^2 - 1).\ 3x^2$

$= 6x^5 - 4x^3 + 6x^2$ (on simplification).

Illustration 63. If $y = 10^x x^{10}$, find $\frac{dy}{dx}$.

Solution:

Let $y = uv$ where $u = 10^x$ and $v = x^{10}$

Now $\frac{du}{dx} = \frac{d}{dx}(10^x) = (10^x)\log_e 10$;

Again $\frac{dv}{dx} = \frac{d}{dx}.(x^{10}) = 10.x^{10-1} = 10x^9$.

$\therefore \frac{dy}{dx} = \frac{d}{dx}(u.v.) = u\frac{dv}{dx} + v\frac{du}{dx} = 10^x\ 10x^9 + x^{10}.\ 10^x \log_e 10 = 10^x(10x^9 + x^{10}\log_e 10)$

ILLUSTRATION RELATED TO DIVISION FORMULA:

Illustration 64. If $y = \frac{x-1}{x+1}$, find $\frac{dy}{dx}$

Solution:

Let $y = \frac{u}{v}$, where $u = x - 1$, $\frac{du}{dx} = \frac{d}{dx}(x-1) = \frac{d(x)}{dx} - \frac{d}{dx}.(1) = 1 - 0 = 1$

$$v = x + 1, \frac{dv}{dx} = \frac{d}{dx}(x+1) = \frac{d(x)}{dx} + \frac{d}{dx}(1) = 1 + 0 = 1$$

$$\therefore \quad \frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2} = \frac{(x+1).1 - (x-1).1}{(x+1)^2} = \frac{x+1-x+1}{(x+1)^2} = \frac{2}{(x+1)^2}$$

Illustration 65. If $y = \dfrac{(x+1)(2x^2-1)}{x^2+1}$, find $\dfrac{dy}{dx}$.

Solution:

Let $y = \dfrac{u}{v}$, where $u = (x + 1)(2x^2 - 1) = 2x^3 + 2x^2 - x - 1$.

and $\dfrac{du}{dx} = 6x^2 + 4x - 1$,

$v = x^2 + 1$; $\dfrac{dv}{dx} = 2x$

$$\frac{dy}{dx} = \frac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2} = \frac{(x^2+1)(6x^2+4x-1)-(x+1)(2x^2-1).\ 2x}{(x^2+1)^2} = \frac{2x^4+3x^3+2x-1}{(x^2+1)^2}$$

DERIVATIVE OF FUNCTION OF A FUNCTION

A variable y may be a function of a second variable z, which again may be a function of a third variable x.

i.e., $y = z^2 + 3$, and $z = 2x + 1$

Here y is a function of z and z again a function of x. Ultimately y is seen to depend on x, so y is called the *function of another function*.

Symbolically, if $y = f(z)$, $z = \varphi(x)$ then $y = f\{\varphi(x)\}$

Theorem. If $y = f(z)$ and $z = \varphi(x)$ then $\dfrac{dy}{dx} = \dfrac{dy}{dz}.\dfrac{dz}{dx}$ (proof is not shown at present)

Corr. If $u = f(v)$, $v = \varphi(w)$, $w = \psi(x)$ then $\dfrac{du}{dx} = \dfrac{du}{dv}.\dfrac{dv}{dw}.\dfrac{dw}{dx}$

Illustration 66. To find $\dfrac{dy}{dx}$ for $y = 2z^2 + 1$, $z = 4x - 2$

Solution:

Now $\dfrac{dy}{dz} = 4z$ and $\dfrac{dz}{dx} = 4$

$$\frac{dy}{dx} = \frac{dy}{dz}.\frac{dz}{dx} = 4z.\ 4 = 16z = 16\,(4x-2) = 64x - 32.$$

Rule 1. If y = ax + b, to find $\frac{dy}{dx}$. Let y = z, and z = ax + b.

So y = f (z) and z = f (x)

$\therefore \quad \frac{dy}{dx} = \frac{dy}{dz}.\frac{dz}{dx} = 1.\ (a.\ 1 + 0) = a$

Rule. 2. If $y = (ax + b)^n$, to find $\frac{dy}{dx}$ Let $y = z^n$ and z = ax + b

Now $\frac{dy}{dz} = n.z^{n-1}$ and $\frac{dz}{dx} = a$

$\therefore \quad \frac{dy}{dx} = \frac{dy}{dz}.\frac{dz}{dx} = nz^{n-1}\ a = na\ (ax + b)^{n-1}$

Illustration 67. : If $y = (2x + 5)^4$ find $\frac{dy}{dx}$

Solution:

Let $y = z^4$, where z = 2x + 5.

Now $\frac{dy}{dz} = 4z^3, \frac{dz}{dx} = 2$

Now $\frac{dy}{dx} = \frac{dy}{dz}.\frac{dz}{dx} = 4z^3.\ 2 = 4.2(2x+5)^3 = 8(2x+5)^3$

Rule 3. If y = log u (u is a function of x), then to find $\frac{dy}{dx}$; $\frac{dy}{dx} = \frac{dy}{du}.\frac{du}{dx}$

Illustration 68. y = log (4x), find $\frac{dy}{dx}$

Solution:

Let y = log u, where u = 4x, $\frac{du}{dx} = 4$

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = \frac{d}{du}(\log u).\ 4 = \frac{1}{u}\cdot 4 = 4.\frac{1}{4x} = \frac{1}{x}.$$

Illustration 69. $y = \log\left(1+\sqrt{x}\right)$, find $\frac{dy}{dx}$

Solution:

Let y = log u, where u = $1+\sqrt{x}$

$$\frac{du}{dx} = 0 + \frac{1}{2\sqrt{x}} = \frac{1}{2\sqrt{x}}\ ;\ \frac{dy}{dx} = \frac{dy}{du}.\frac{du}{dx} = \frac{1}{u}.\frac{1}{2\sqrt{x}} = \frac{1}{\left(1+\sqrt{x}\right).\ 2\sqrt{x}}$$

Illustration 70. If $y = \sqrt{x^2+7}$, find $\frac{dy}{dx}$.

Solution: Let y $= \sqrt{z}$, where $z = x^2 + 7$, $\frac{dz}{dx} = 2x$.

$$\frac{dy}{dx} = \frac{dy}{dz}.\frac{dz}{dx} = \frac{d}{dz}.\ \sqrt{z}.\ 2x = \frac{1}{2\sqrt{z}}.\ 2x = \frac{x}{\sqrt{z}} = \frac{x}{\sqrt{x^2+7}}$$

Illustration 71. Given $y = \sqrt{\frac{2x+1}{x+2}}$, find $\frac{dy}{dx}$.

Solution:

Let $y = \sqrt{u}$, where $u = \frac{2x+1}{x+2}$, $\frac{du}{dx} = \frac{(x+2)\frac{d}{dx}(2x+1) - (2x+1)\frac{d}{dx}(x+2)}{(x+2)^2}$

$$\frac{du}{dx} = \frac{(x+2).2 - (2x+1).1}{(x+2)^2} = \frac{3}{(x+2)^2}$$

$$\frac{dy}{du} = \frac{1}{2\sqrt{u}} = \frac{1}{2}.\ \sqrt{\frac{x+2}{2x+1}}$$

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = \frac{1}{2}\sqrt{\frac{x+2}{2x+1}}.\frac{3}{(x+2)^2} = \frac{3}{2\sqrt{2x+1}.\ (x+2)^{3/2}}$$

Illustration 72. If y = log log log x^2, find $\frac{dy}{dx}$

Solution:

Let y = log u where u = log v and v = log x^2 = 2 logx.

$$\frac{dy}{du} = \frac{1}{u} = \frac{1}{\log v} = \frac{1}{\log\log x^2}.$$

$$\frac{du}{dv} = \frac{1}{v} = \frac{1}{\log x^2} = \frac{1}{2\log x}.$$

$$\frac{dv}{dx} = \frac{2}{x}$$

$$\therefore\quad \frac{dy}{dx} = \frac{dy}{du}.\frac{du}{dv}.\frac{dv}{dx} = \frac{1}{\log\log x^2}.\frac{1}{2\log x}.\frac{2}{x} = \frac{1}{x\log x\log\log x^2}$$

Illustration 73. Differentiate x^5 w.r.t.x^2

Solution:

Let $y = x^5$, $z = x^2$. So the requirement of the question is $\frac{d}{d(x^2)}(x^5) = \frac{dy}{dz}$

$\frac{dy}{dx} = 5x^4, \frac{dz}{dx} = \frac{d}{dx}(x^2) = 2x,$ so that $\frac{dx}{dz} = \frac{1}{2x}.$

Now $\quad \frac{dy}{dz} = \frac{dy}{dx}.\frac{dx}{dz} = 5x^4.\frac{1}{2x} = \frac{5}{2}x^3.$

Illustration 74. $y = \log_e\left(x + \sqrt{x^2 + a^2}\right)$, find $\frac{dy}{dx}$.

Solution:

Let $y = \log u$ where $u = x + \sqrt{x^2 + a^2}$

$\frac{dy}{du} = \frac{1}{u}$ and $\frac{du}{dx} = \frac{d}{dx}(x) + \frac{d}{dx}\left(x^2 + a^2\right)^{1/2}$ or, $\frac{du}{dx} = 1 + \frac{1}{2\sqrt{x^2 + a^2}}.2x$

Now $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = \frac{1}{u}.\left(1 + \frac{x}{\sqrt{x^2 + a^2}}\right) = \frac{1}{x + \sqrt{x^2 + a^2}}.\frac{x + \sqrt{x^2 + a^2}}{\sqrt{x^2 + a^2}} = \frac{1}{\sqrt{x^2 + a^2}}.$

DERIVATIVE OF IMPLICIT FUNCTION

If f(x, y) = 0 defines y as a derivable function of x, then differentiate each term w.r.t.x. The idea will be clear from the given **Illustration**.

Illustration 75. Find $\frac{dy}{dx}$, if $3x^4 - x^2y + 2y^3 = 0$

Solution:

Differentiating each term of the functions w.r.t.x we get, $3.4x^3 - \left(x^2\frac{dy}{dx} + 2xy\right) + 6y^2\frac{dy}{dx} = 0$

or, $\quad 12x^3 - x^2\frac{dy}{dx} - 2xy + 6y^2\frac{dy}{dx} = 0$

or, $\quad \left(6y^2 - x^2\right)\frac{dy}{dx} = 2xy - 12x^3 \qquad$ or, $\frac{dy}{dx} = \frac{2xy - 12x^3}{6y^2 - x^2}.$

DERIVATIVE OF PARAMETRIC FUNCTION

When each of the variables x and y can be expressed in terms of a third variable (say t) then the function y = f(x) is known as parametric function.

For example. $x = f_1(t), y = f_2(t)$. Now to find $\frac{dy}{dx}$ we are to find $\frac{dy}{dt}$ and $\frac{dx}{dt}$ so that $\frac{dy}{dx} = \frac{dy}{dt}.\frac{dt}{dx} = \frac{dy/dt}{dx/dt}, \frac{dx}{dt} \neq 0.$

Illustration 76. Find $\frac{dy}{dx}$ when x = 4t, y = 2t^2

Solution:

$$\frac{dx}{dt} = 4, \frac{dy}{dt} = 4t, \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{4t}{4} = t$$

Illustration 77. Find $\frac{dy}{dx}$, when $x = \frac{3\,at}{1+t^3},\ y = \frac{3at^2}{1+t^3}$

Solution:

$$\frac{dx}{dt} = \frac{(1+t^3).3a - 3at(3t^2)}{(1+t^3)^2} = \frac{3a(1-2t^3)}{(1+t^3)^2}$$

$$\frac{dy}{dt} = \frac{(1+t^3)\,6at - 3at^2(3t^2)}{(1+t^3)^2} = \frac{3at\,(2-t^3)}{(1+t^3)^2}$$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{(1+t^3)\,6at - 3at^2(3t^2)}{(1+t^3)^2} + \frac{3a(1-2t^3)}{(1+t^3)^2} = \frac{3at\,(2-t^3)}{3a\,(1+2t^3)} = \frac{t(2-t^3)}{1-2t^3}$$

SECOND ORDER DERIVATIVE

Introduction :

We have seen that the first order derivative of a function of x, say f(x), may also be a function of x. This new function of x also may have a derivative w.r.t.x which is known as second order derivative of f(x) i.e. second order derivative is the derivative of first order derivative.

Similarly the derivative of the second order derivative is known as third order derivative and so on up to *n*th order.

Symbols :

For the function y = f(x), the first order derivative w.r.t.x is denoted by $\frac{dy}{dx}$ or f′ (x) or y_1 as discussed before.

Now the second order derivative of y = f(x) is expressed as $\frac{d^2y}{dx^2}$ or f ″ (x) or y_2. The notation $\frac{d^2y}{dx^2}$ is read as "*dee two y by dee x squared*".

Illustration 78. If y = x^4, find $\frac{d^2y}{dx^2}$.

Solution:

$$y = x^4, \frac{dy}{dx} = 4x^3, \text{again} \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}(4x^3) = 4.3x^{3-1} = 12x^2.$$

Note : To find $\frac{d^3y}{dx^3}$; $\frac{d^3y}{dx^3} = \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) = \frac{d}{dx}\left(12x^2\right) = 12.\frac{d}{dx}(x^2)$

= 12.2x$^{2-1}$ = 24x i.e., third order derivative is 24x.

Illustration 79. Find $\frac{d^2y}{dx^2}$ if $y = \frac{\log x}{x}$.

Solution:

$$\frac{dy}{dx} = y_1 = \frac{x.\frac{1}{x} - \log x.1}{x^2} = \frac{1 - \log x}{x^2};$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{1-\log x}{x^2}\right) = \frac{x^2\left(-\frac{1}{x}\right) - (1 - \log x).\,2x}{x^4}$$

$$= \frac{-x - 2x(1-\log x)}{x^4} = \frac{-1-2\,(1-\log x)}{x^3} = \frac{2\log x - 3}{x^3}.$$

Illustration 80.

If $y = x^2 e^x$, Prove $\frac{d^2y}{dx^2} = (x^2 + 4x + 2)e^x$

Solution:

$y = x^2 e^x$

Differentiating both sides w.r.t x

$$\frac{dy}{dx} = x^2.e^x + e^x.2x$$

$= (x^2 + 2x)e^x$

Again differentiating both sides w.r.t x

$$\frac{d^2y}{dx^2} = (x^2 + 2x)\, e^x + e^x\,(2x + 2)$$

$= (x^2 + 2x + 2x + 2)e^x$

$= (x^2 + 4x + 2)e^x$

FOR IMPLICIT FUNCTION AND PARAMETRIC FORMS :

Illustration 81. For $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, find $\frac{d^2y}{dx^2}$

Solution:

Diff. Both sides w.r.t. x we get $\frac{2x}{a^2} + \frac{2y.y_1}{b^2} = 0$ or $\frac{2x}{a^2} + \frac{2y}{b^2}.y_1 = 0$ or $y_1 = -\frac{b^2}{a^2}.\frac{x}{y}$

$$\frac{d^2y}{dx^2} = \left(-\frac{b^2}{a^2}\right).\frac{y.1 - x.y_1}{y^2} = \left(-\frac{b^2}{a^2}\right)\frac{y + x\left(\frac{b^2}{a^2}.\frac{x}{y}\right)}{y^2}$$ [putting the value of y_1]

$$= \left(-\frac{b^2}{a^2}\right).\frac{(a^2y^2 + b^2x^2)}{a^2y^3} = \left(-\frac{b^2}{a^2}\right).\frac{a^2b^2}{a^2y^3}$$ [from the given equation we have $a^2y^2 + b^2x^2 = a^2b^2$]

$$= -\frac{b^4}{y^3}$$

Illustration 82.

If $y = t^2 + t^3$, $x = t - t^4$, find $\frac{d^2y}{dx^2}$.

Solution:

$$\frac{dy}{dt} = 2t + 3t^2, \frac{dx}{dt} = 1 - 4t^3$$

$$\frac{dy}{dx} = \frac{dy}{dt}.\frac{dt}{dx} = \frac{dy}{dt} \Big/ \frac{dx}{dt} = \frac{2t + 3t^2}{1 - 4t^3}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dt}\left(\frac{dy}{dx}\right).\frac{dt}{dx} = \frac{d}{dt}\left(\frac{dy}{dx}\right) \Big/ \frac{dx}{dt} = \frac{d}{dt}\left(\frac{2t + 3t^2}{1 - 4t^3}\right) \Big/ \left(1 - 4t^3\right)$$

$$= \frac{\left(1 - 4t^3\right)\left(2 + 6t\right) - \left(2t + 3t^2\right)\left(-12t^2\right)}{\left(1 - 4t^3\right)^2 . \left(1 - 4t^2\right)} = \frac{12t^4 + 16t^3 + 6t + 2}{\left(1 - 4t^3\right)^2}$$

Illustration 83. Find gradient to the curve $y = \sqrt{4 - x^2}$ at the point. Where the ordinate & abscissa are same.

Solution:

$y = \sqrt{4 - x^2}$

Or, $\frac{dy}{dx} = \frac{-2x}{2\sqrt{4 - x^2}} = -\frac{x}{y} = -\frac{x}{x} = -1.$

Illustration 84. If $f(x) = x^k$. and $f'(1) = 10$, find 'k'.

Solution:

$f'(x) = Kx^{K-1}$

Or $f'(1) = K = 10.$

Illustration 85. If $f(x) = x^2 - 6x + 8$, find $f'(5) - f'(2)$.

Solution:

$f'(x) = 2x - 6$

$f'(5) = 10 - 6 = 4$

$f'(2) = 4 - 6 = -2$

$f'(5) - f'(2) = 4 - (-2) = 6.$

Illustration 86. If $f(x) = xe^x$, find x if $f'(x) = 0$

Solution:

$f'(x) = x . \frac{d(e)^x}{dx} + e^x \frac{dx}{dx} = 0$

Or, $xe^x + e^x = 0$

Or, $e^x (x+1) = 0$ or $x+1 = 0$ or $x = -1$

Illustration 87. If $f(x) = \frac{x}{e^x}$, find x if $f`(x) = 0$.

Solution:

$$f`(x) = \frac{e^x\frac{dx}{dx} - x\frac{d(e^x)}{dx}}{e^{2x}} = 0$$

Or, $e^x - xe^x = 0$

Or, $e^x(1 - x) = 0$ or $1 - x = 0$ or $1 = x$

Illustration 88. If ${}^xc_2 = f(x)$, find $f`(1)$.

Solution:

$$f`(x) = \frac{x!}{2!(x-2)!} = \frac{x(x-1)}{2}.$$

$$f`(x) = \frac{1}{2}\frac{d(x^2 - x)}{dx} = \frac{1}{2}(2x - 1)$$

$$f`(1) = \frac{1}{2}(2-1) = \frac{1}{2}.$$

Illustration 89. Find gradiant of the curve $y = \log^x_e$ at (1,0).

Solution:

$y = \log^x_e$ Or $0 = \log^x_e$

Or, $\log^1_e = \log^x_e$ Or, $1 = x$.

Or, $\frac{dy}{dx} = \frac{1}{x} = \frac{1}{1} = 1.$

Illustration 90. If $xy = 1$, find $y^2 + f`(x)$.

Solution:

$$y = \frac{1}{x}$$

Or, $f`(x) = -\frac{1}{x^2}$

So, $y^2 + f`(x) = \frac{1}{x^2} - \frac{1}{x^2} = 0.$

Illustration 91. Find gradiant of the curve $y = 3x^2 - 5x$ at (1, 2).

Answer:

$f`(x) = 6x - 5$ at $x = 1$

$= 6 - 5 = 1.$

Illustration 92. $y = 2^{\log_2^x}$, find $f`(x)$.

Answer:

$y = x.$

Or, $f`(x) = \frac{dx}{dx} = 1.$

Revenue and Cost Function

3.2

Revenue

The term 'revenue' refers to the receipts obtained by a firm form the sale of certain quantities of a commodity at various prices. The revenue concept relates to total revenue, average revenue and marginal revenue.

- **Total Revenue**: Total revenue refers to the total amount of money that the firm receives from the sale of its products. Thus, total revenue is equal to the quantity sold multiplied by the selling price of the commodity. Algebraically total revenue can be written as:

$$TR = PQ$$

where, TR = Total Revenue; P = Price per unit; Q = Quantity sold

- **Average Revenue:** Average Revenue can be obtained by dividing the total revenue by the number of units sold. Thus, Average Revenue can be written as:

$$Average\ Revenue = \frac{Total\ Revenue}{Total\ output\ sold}$$

Or, $$AR = \frac{TR}{Q} = \frac{PQ}{Q} = P$$

Thus, average revenue is the price of the commodity.

- **Marginal Revenue:** Marginal Revenue is the net revenue earned by selling one additional unit of the product produced. In other words, marginal revenue is the addition made to the total revenue by selling one additional unit of the commodity. Putting it in algebraic expression, marginal revenue is the addition to total revenue by selling $(n+1)$ units of a product instead of n units. Therefore,

Marginal Revenue = difference in total revenue in increasing sales from $(n + 1)$ units to n Units

$$MR = TR_{n+1} - TR_n$$

In other words Marginal Revenue is the rate of change of Total Revenue with respect to the Quantity sold.

If TR stands for Total Revenue and Q stands for output, the marginal revenue (MR) can be expressed as:

$$MR = \frac{d(TR)}{dQ}$$

Here, $\frac{d(TR)}{dQ}$ indicates derivative of TR with respect to Q which is also the slope of the total revenue curve. Thus, if total revenue is given to us, we can find out marginal revenue at various levels of output by finding the slopes at the corresponding points on the total revenue curve.

Illustrations 93.

A schedule on Total Revenue and Amount of output sold is given. Find out Average Revenue and Marginal Revenue

Quantity of Output sold (units)	Total Revenue (TR)	Average Revenue	Marginal Revenue
0	0		
1	29		
2	54		
3	72		
4	80		
5	80		
6	72		

Solution:

Quantity of Output sold (units)	Total Revenue (TR)	Average Revenue $\left(\frac{TR}{Q} = \frac{PQ}{Q} = P\right)$	Marginal Revenue $(MR = TR_{n+1} - TR_n)$
0	0	0	-
1	29	29	29
2	54	27	25
3	72	24	18
4	80	20	8
5	80	16	0
6	72	12	-8

Illustrations 94. Consider the following market demand schedule:

Price per unit (₹)	3	5	8	12	15	20
Quantity (unit)	25	21	18	15	10	4

Now determine the AR and TR schedules of the seller

Solution:

Quantity of Output sold (Q) (units)	Price per unit (P) (₹)	Total Revenue (TR=P.Q) (₹)	Average Revenue (₹) $\left(\frac{TR}{Q} = \frac{PQ}{Q} = P\right)$
25	3	75	3
21	5	105	5
18	8	144	8
15	12	180	12
10	15	150	15
4	20	80	20

Cost

The term 'total cost' is the sum of all costs incurred by a firm in producing a certain level of output. In the traditional theory total cost is divided into two groups: **total fixed cost and total variable cost.**

$TC = TFC + TVC$

where, TC = Total Cost; TFC = Total Fixed Cost; TVC = Total Variable Cost

- **Total Fixed cost:** There are some cost which do not vary with the level of output produced. These remain fixed even when output changes. These are fixed costs. For eg, rent of land, license fees etc. Suppose, that the land on which a factory is situated has an annual rent of ₹ 12,000. It is obvious that the amount of rent will not have any relationship with the quantity of output produced by the factory. Whether the output is 20 tons or 2,000 tons, the rent will remain fixed at ₹ 12,000. In fact, the rent would be the same, even if the factory is temporarily closed down. For this reason, the rent on land is a fixed cost.
- **Total Variable Cost**: On the other hand, some costs change with the level of output produced. This type of cost rises when output increase, and falls when output decrease. For eg, cost of raw materials, wages and salaries of casual employees, transport cost etc. Suppose in order to produce a huge amount of output, we require higher amount of raw materials. So, the cost of raw materials will increase and hence variable cost will increase.

Average Cost: The average cost is obtained by dividing Total cost by the level of output. Thus,

$$Average\ Cost\ (AC) = \frac{Total\ Cost\ (TC)}{Output}$$

Similarly, $$Average\ Fixed\ Cost\ (AFC) = \frac{Total\ Fixed\ Cost\ (TFC)}{Output}$$

$$Average\ Variable\ Cost\ (AVC) = \frac{Total\ Variable\ Cost\ (TVC)}{Output}$$

Marginal Cost: The marginal cost is defined as the change in total cost which results from a unit change in output. In other words, marginal cost is the addition to the total cost when the quantity of output produced increases by one unit. In symbols it can be written as:

$$MC = TC_{n+1} - TC_n$$

Mathematically, the marginal cost is the first order derivative of the Total cost function. Denoting total cost by TC and output by Q we have:

$$MC = \frac{d(TC)}{dQ}$$

It is worth noting that marginal cost is independent of fixed cost. Since fixed cost do not change with the level of output, there is no marginal fixed cost when output is increased.

Profit: The term 'profit' is defined as the difference between total revenue and total cost. Thus,

$$Profit = Total\ Revenue - Total\ Cost$$

In order to obtain the maximum amount of output we differentiate the profit function with respect to the output produced and set the equation equal to zero (First order condition for maximization). Then, we find the second order derivative of the function. If the second order derivative is less than zero then we can conclude that the profit is maximized at that level of output. Thus,

$$\frac{d(Profit)}{dQ} = 0 \qquad \text{[First Order Condition for maximisation]}$$

$$\frac{d^2(Profit)}{dQ^2} < 0 \qquad \text{[Profit is maximized at } Q_{max}\text{]}$$

So for mamimum Profit, $\frac{d(Profit)}{dQ} = 0 \Rightarrow \frac{d\,(TR - TC)}{dQ} = 0 \Rightarrow \frac{d\,(TR)}{dQ} - \frac{d\,(TC)}{dQ} = 0 \Rightarrow MR - MC = 0 \Rightarrow \mathbf{MR = MC}$

Thus profit is maximum at the equilibrium point under perfect competition. In other words maximum profit is obtained when Marginal Cost and Marginal Revenue are equal.

Break-even Point: Break-even point is the point at which total cost and total revenue are equal which means that the firm does neither earn any profit nor any loss. The firm faces a no gain no loss situation. Thus, at the *Break – even Point* :

Profit = 0

Or, *Total Cost* = *Total Revenue*

Illustrations 95.

For a firm the Total Revenue (TR) and Total Cost (TC) functions are given as TR = 20Q and TC = Q2 + 4Q + 20 where Q = Output. Find the profit maximizing output and maximum profit.

Solution:

Profit = TR-TC

$= 20Q - Q^2 - 4Q - 20$

$= 16Q - Q^2 - 20$

$$\frac{d\,(Profit)}{dQ} = 16 - 2Q$$

$\therefore\ 16 - 2Q = 0$ [Find Order Condition for maximization]

Or, $Q = 8$

$$\frac{d^2(Profit)}{dQ^2} = \frac{d}{dQ}\left\{\frac{d\,(Profit)}{dQ}\right\} = \frac{d}{dQ}(16 - 2Q) = -2 < 0$$

So, the profit is maximized at $Q = 8$

The maximum profit = $16 \times 8 - 8^2 - 20 = 44$

Illustrations 96. In a firm, price of a product is ₹4 and TC = Q3 – 15Q2 + 31Q + 100 where Q = Output. Find the profit maximizing output and maximum profit

Solution:

Total Revenue = Price × Quantity = $4Q$

Profit = TR-TC

$= 4Q - Q^3 + 15Q^2 - 31Q - 100$

$$= -27Q - Q^3 + 15Q^2 - 100$$

$$\frac{d\ (Profit)}{dQ} = -27 - 3Q^2 + 30Q$$

$\therefore 3Q^2 - 30Q + 27 = 0$ [First Order Condition for maximization]

Or, $Q^2 - 10Q + 9 = 0$

Or, $Q^2 - Q - 9Q + 9 = 0$

Or, $Q(Q-1) - 9(Q-1) = 0$

Or, $(Q-9)(Q-1) = 0$

Or, $Q = 9, 1$

$$\frac{d^2(Profit)}{dQ^2}\ \frac{d}{dQ}\left[\frac{d\ (\text{Profit})}{dQ}\right] = \frac{d}{dQ}[-27 - 3Q^2 + 30Q] = -6Q + 30$$

At $Q = 1$, Profit = 24 > 0

At $Q = 9$, Profit = – 24 < 0

So, the profit is maximized at $Q = 9$

The maximum profit at $Q = 9$ is $(-27) \times 9 - 9^3 + 15 \times 9^2 - 100 = 143$

Illustrations 97. A firm has the following functions of Total Revenue (TR) and Total Cost (TC) for Q output TR = 30 Q – Q^2 and TC = (Q^3 –15Q^2 +10Q +100). Find the profit maximizing output and maximum profit and equilibrium price.

Solution:

Profit = TR–TC

$$= 30Q - Q^2 - Q^3 + 15Q^2 - 10Q - 100$$

$$= 20Q - Q^3 + 14Q^2 - 100$$

$$\frac{d\ (Profit)}{dQ} = 20 - 3Q^2 + 28Q$$

$\therefore 3Q^2 - 28Q - 20 = 0$ [F.O.C for maximization]

Or, $Q = \dfrac{28 \pm \sqrt{28^2 + 4 \times 3 \times 20}}{2 \times 3}$ [by using Sridhar Acharya Formula]

Or, $Q = \dfrac{28 \pm \sqrt{784 + 240}}{6}$

Or, $Q = \dfrac{28 \pm 32}{6}$

Or, $Q = 10, -\dfrac{2}{3}$

$$\frac{d^2(Profit)}{dQ^2} = \frac{d}{dQ}\left[\frac{d\,(Profit)}{dQ}\right] = \frac{d}{dQ}[20-3Q^2+28Q] = -\ 6Q + 28$$

At $Q = 10$, Profit = – 32 < 0

At $Q = -\frac{2}{3}$, Profit = 32 > 0

So, the profit is maximized at $Q = 10$

The maximum profit at $Q = 10$ is $20 \times 10 - 10^3 + 14 \times 10^2 - 100 = 500$

We know that P = AR, so $AR = \frac{TR}{Q} = 30 - Q$

As profit is maximized at $Q = 10$ then $P = 30 - 10 =$ ₹ 20 is the equilibrium price

Illustrations 98. A firm has the following functions of Total Revenue (TR) and Total Cost (TC) for Q output TR = 26Q – 3Q2 and TC = 2Q2 – 4Q + 10. Find the profit maximizing output and maximum profit and equilibrium price and Total Revenue at that level of output.

Solution:

Profit = $TR - TC$

$$= 26Q - 3Q^2 - 2Q^2 + 4Q - 10$$

$$= 30Q - 5Q^2 - 10$$

$$\frac{d(Profit)}{dQ} = 30 - 10Q$$

$\therefore 30 - 10Q = 0$ [F.O.C for maximization]

Or, $Q = 3$

$$\frac{d^2(Profit)}{dQ^2} = \frac{d}{dQ}\left[\frac{d\,(Profit)}{dQ}\right] = \frac{d}{dQ}[30-10Q] = -10 < 0$$

So, the profit is maximized at $Q = 3$

The maximum profit at $Q = 3$ is $30 \times 3 - 5 \times 3^2 - 10 = 35$

We know that P = AR, so $AR = \frac{TR}{Q} = 26 - 3Q$

As profit is maximized at $Q = 3$ then $P = 26 - 3 \times 3 = 17$ Money units. This is the equilibrium price

Total Revenue at this price = $26 \times 3 - 3 \times 3^2 = 51$ Money units

Optimization Techniques (Basic Concepts)

3.3

Slope and Curvature

For a differentiable function $y = f(x)$ the derivative $\frac{dy}{dx}$ or $f'(x)$ is often called the first derivative or the first order derivative. Generally this first order derivative $f'(x)$ is a function of x and therefore $f'(x)$ is further differentiable with respect to x which is called the second order derivative of the function $y = f(x)$. This second order derivative is denoted by $f''(x)$ or $\frac{d^2y}{dx^2}$. In the same manner if $f''(x)$ be differentiable then we can obtain the third order derivative $f'''(x)$ or $\frac{d^3y}{dx^3}$. Thus, proceeding in the similar way one can introduce the n^{th} order derivative of $y = f(x)$ which will be denoted by $\frac{d^ny}{dx^n}$.

The first and second order derivatives are much used in the graphical analysis. First derivative of a function at any point measures the slope of a function at a particular point. If $f'(x) > 0$ for all values of x in the domain of the function then we can say that $f(x)$ rises continuously from left to right. But we know that the function which rises continuously is called monotonically increasing function. Thus, the condition of a function to be monotonically increasing is that the first derivative must be positive all through. Similarly, a function $y = f(x)$ is said to be monotonically decreasing is $f'(x) < 0$ for all x. In short the condition of monotonicity for a function $y = f(x)$ is either $f'(x) > 0$ or $f'(x) < 0$ for all x.

Illustrations 99.

1. $f(x) = 2x$

   $f'(x) = 2 > 0$ This function is monotonically increasing

2. $f(x) = -6x$

   $f'(x) = -6 < 0$. This function is monotonically decreasing

To know whether a function is increasing at a constant rate or at an increasing rate or at a decreasing rate, we perform the second order derivative. The nature of the function (constant, increasing, decreasing) can be known form the knowledge of the curvature of the curve. By curvature we mean the bending of a curve. To measure this bending we have to measure the rate of change of the slope of the curve. If the rate of change of the slope of the curve is positive i.e. $f''(x) > 0$ the curve is said to be convex curve. If $f'(x) > 0$ and $f''(x) > 0$ the curve increases at an increasing rate. On the other hand, if $f'(x) < 0$ and $f''(x) > 0$ the curve decreases at an increasing rate. Lastly, there may be some curve which increase or decreases at a constant rate. Linear curves are examples of this case and for this $f''(x) = 0$.

Now, we can also identify whether a curve is convex or concave by just looking at the curve and by drawing a tangent at any point on the curve. If the curve lies above the tangent it will be convex at that point, otherwise if the curve lies entirely below the tangent it will be concave at that point.

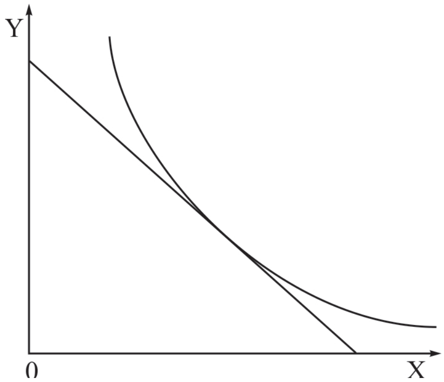


Fig. 1

Convex

$f'(x) < 0$, $f''(x) > 0$

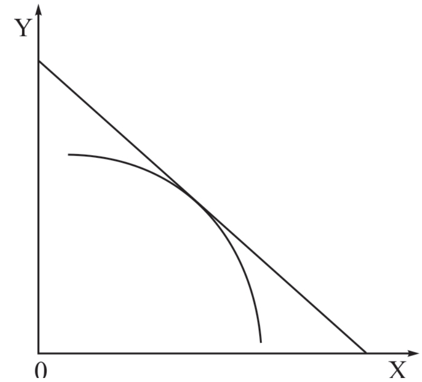


Fig. 2

Concave

$f'(x) < 0$, $f''(x) < 0$

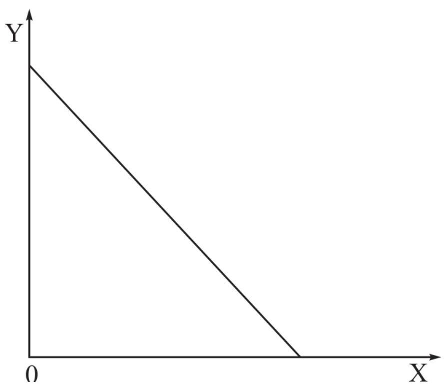


Fig. 3

$f'(x) < 0$, $f''(x) = 0$

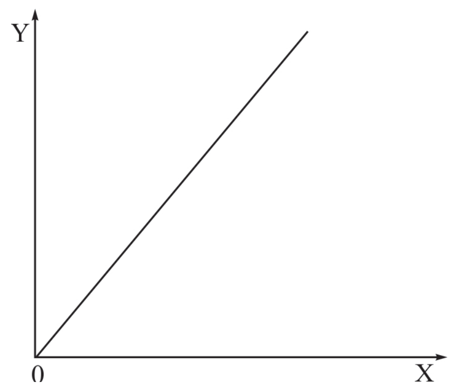


Fig. 4

$f'(x) > 0$, $f''(x) = 0$

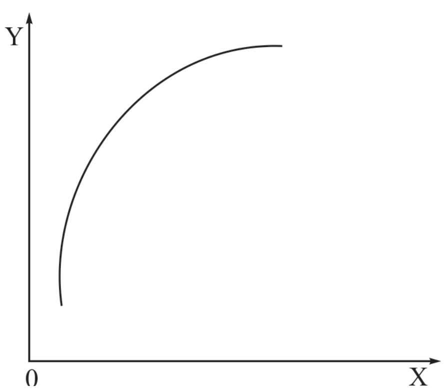


Fig. 5

$f'(x) > 0$, $f''(x) < 0$

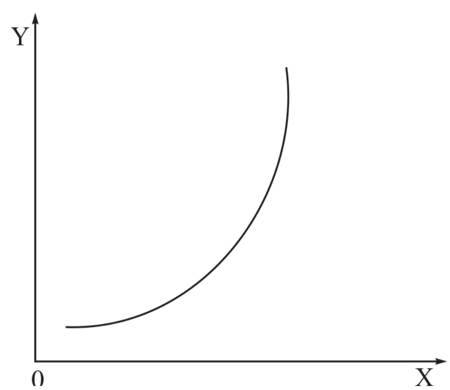


Fig. 6

$f'(x) > 0$, $f''(x) > 0$

Maxima and Minima

In order to determine the relative extremum (relative maximum and minimum) of a function $y = f(x)$, the first derivative plays an important role.

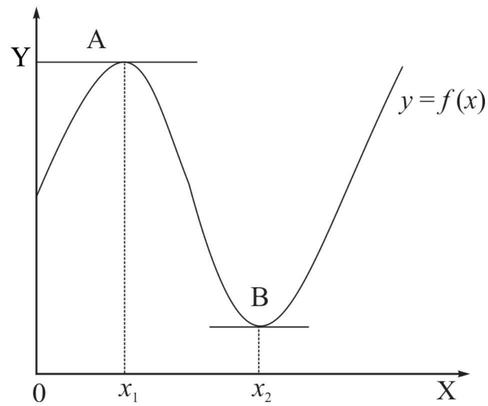


Fig. 7

In the figure 7 given here the function $y = f(x)$ is attaining the maximum value at the point A, where $x = x_1$ and minimum value at the point B, where $x = x_2$. It appears that the slope of the curve is positive before it attains maximum at the point $x = x_1$ and is negatively sloped immediately after attaining the maximum value and at the maximum point A the slope of the curve is zero as the tangent at A is horizontal. Since the slope of a function $y = f(x)$ is represented by its first derivative $\frac{dy}{dx}$, so the necessary condition for maximization is $\frac{dy}{dx} = f'(x) = 0$. Similarly it appears that the curve is sloping downward before attaining minimum value at B and immediately after attaining the minimum it slopes upward indicating that the slope of the curve is zero at the minimum point B since the tangent at B is horizontal. Since, the slope of the curves at both the extreme point are 0, first order derivative is set at 0 is a necessary condition for both maximization and minimization.

For sufficient condition for maximum and minimum value of the function $y = f(x)$, we have to see the slope of the curve immediately after attaining the maximum or minimum. From the figure, it is seen that after attaining maximum point at A the curve is sloping downward while it is sloping upwards after attaining minimum. Thus, a distinction between maximum and minimum can be made by examining the direction of change of the first derivative. Given the function $y = f(x)$ whose first derivative $f'(x) = 0$ at point $x = x_1$ will be a relative maximum if the derivative $f'(x)$ changes its sign from positive to negative as x increases in the neighbourhood of $x = x_1$. Similarly, $y = f(x)$ will attain a relative minimum if the derivative $f'(x)$ changes its sign from negative to positive as x increases in the neighbourhood of $x = x_2$. Since the direction of change of first derivative can be stated in terms of the sign of the second derivative $f''(x)$, the sufficient condition for maximization is $f''(x) < 0$ and for minimization is $f''(x) > 0$.

Illustrations 100.

If $y = 20x - 2x^2$ then find the value of x for which y is optimum.

Solution:

$\frac{dy}{dx} = 20 - 4x$.

As $\frac{dy}{dx} = 0$ according to the first order condition,

$20 - 4x = 0$ or, $4x = 20$ or, $x = 5$

$\frac{d^2y}{dx^2} = -4 < 0$ So, the function is maximum at $x = 5$

Illustrations 101. If $y = 2x^2 - 16x + 50$, then find x where y is optimum.

Solution:

$\frac{dy}{dx} = 4x - 16$.

As $\frac{dy}{dx} = 0$ according to the first order condition,

$4x – 16 = 0$ or, $4x = 16$ or, $x = 4$

$\frac{d^2y}{dx^2} = 4 > 0$ So, the function is minimum at $x = 4$

Illustrations 102. If $y = x^3 - 9x^2 + 15x + 20$, then find the points where y has maximum and minimum values

Solution:

$\frac{dy}{dx} = 3x^2 - 18x + 15$.

As $\frac{dy}{dx} = 0$ according to the first order condition,

$3x^2 - 18x + 15 = 0$

Or, $3x^2 - 15x - 3x + 15 = 0$

Or, $3x(x-5) - 3(x-5) = 0$

Or, $(3x-3)(x-5) = 0$ or, $x = 1,5$

$\frac{d^2y}{dx^2} = 6x - 18$

Putting $x = 1$, we get $\frac{d^2y}{dx^2} = -12 < 0$

Putting $x = 5$, we get $\frac{d^2y}{dx^2} = 12 > 0$

So, the function is maximum at $x = 1$ and minimum at $x = 5$

Illustrations 103. Find x where $y = \frac{1}{3}x^3 - \frac{5}{2}x^2 + 4x + 10$ has optimum values.

Solution:

$\frac{dy}{dx} = x^2 - 5x + 4$.

As $\frac{dy}{dx} = 0$ according to the first order condition,

$x^2 - 5x + 4 = 0$

Or, $x^2 - x - 4x + 4 = 0$

Or, $x(x-1) - 4(x-1) = 0$

Or, $(x-4)(x-1) = 0$ or, $x = 1,4$

$\frac{d^2y}{dx^2} = 2x - 5$

Putting $x = 1$, we get $\frac{d^2y}{dx^2} = -3 < 0$

Putting $x = 4$, we get $\frac{d^2y}{dx^2} = 3 > 0$

So, the function is maximum at $x = 1$ and minimum at $x = 4$

Exercise:

Theoretical Questions

Multiple Choice Questions (MCQ)

1. $\lim_{x\to 3}(x^3+1)$
   (a) 52
   (b) 53
   (c) 55
   (d) 54

2. $\lim_{x\to 0}(4x^2+7x+5)$
   (a) 4
   (b) 7
   (c) 5
   (d) 16

3. $\lim_{x\to 4}\left(\frac{x^2-16}{x-4}\right)$
   (a) 2
   (b) 4
   (c) 8
   (d) 6

4. If $y = xe^x$ then $\frac{dy}{dx} = ?$
   (a) xe^x
   (b) $e^x(x+1)$
   (c) $e^x(x-1)$
   (d) e^x / x

5. $f(x) = \frac{x^5}{5}+\frac{x^4}{4}+\frac{x^3}{2}-7x^2+18 \cdot f'(x) = ?$
   (a) $\frac{x^4}{4}+\frac{x^3}{3}+\frac{3x^2}{2}+14x$
   (b) $\frac{x^6}{6}+\frac{x^5}{5}+\frac{x^4}{4}-7x^3+18x$
   (c) $x^4-x^3-\frac{3x^2}{2}+14x$
   (d) $x^4+x^3+\frac{3x^2}{2}-14x$

6. When $y = 4^x$ then derivative of y is ——

(a) $x(4^{x-1})$

(b) $\frac{4^x}{2\log 2}$

(c) $4^x\ 2\log 2$

(d) None of these

7. Find the differential coefficient of $y = \frac{e^x}{e^x + 1}$

(a) $\frac{e^x}{(e^x+1)^2}$

(b) $\frac{e^{2x}}{(e^x+1)^2}$

(c) $\frac{e^x}{(e^{2x}+1)^2}$

(d) $\frac{e^x}{(e^x+2)^2}$

8. $y = (4x-3)^3 + (5x-2)^2$. Calculate y_1

(a) $182x^2 + 13x + 29$

(b) $96x^2 + 13x + 29$

(c) $12x^2 + 26x + 29$

(d) $192x^2 + 26x + 58$

9. $\frac{x^2}{16} + \frac{y^2}{4} = 1$ is an Implicit function. The derivative of this function is ——

(a) $\frac{x}{4y}$

(b) $-\frac{x}{4y}$

(c) $\frac{x}{2y}$

(d) $-\frac{x}{2y}$

10. The result of differentiation of $y = \log\left[3x^2 + 13x + 10\right]$ is ——

(a) $\frac{6x^2+13x}{3x^2+13x+10}$

(b) $\frac{6x+13}{3x^3+13x^2+10x}$

(c) $\frac{6x+13}{3x^2+13x+10}$

(d) $\frac{19}{6x+13}$

11. A soft-drink manufacturer has a revenue function $R = 7Q^2 - 19Q + 30$ and the cost function is given by $9Q$. Find the number of cans produced by the firm, under perfect competition.

(a) 2

(b) 4

(c) 6

(d) 8

12. A tin manufacturer has a revenue function given by: $R = 11Q^2 - 110Q + 70$ and the cost function is given by: $C = 22Q$. Find the number of tins to be produced by the manufacturer.

(a) 2

(b) 6

(c) 10

(d) 14

13. A demand function is given by: $P = a - bQ$ and the cost function is given by $C = Q^2$. Find the value of Q for which profit will be maximum under perfect competition.

(a) $\frac{a}{(a+1)}$

(b) $\frac{a}{2(b+1)^2}$

(c) $\frac{a}{2(b+1)}$

(d) $\frac{b}{2(a+1)}$

14. The demand function is given by: $P = 1400 - 25Q$ and the cost function is given by $C = 10Q^2$. Find the value of Q at the equilibrium point.

(a) 10

(b) 20

(c) 30

(d) 40

15. A revenue function is given by $R = 3Q^2 - 8Q + 15$ and the cost function is given by: $C = 28Q$. Find the value of Q for achieving highest profit.

(a) 3

(b) 9

(c) 6

(d) 12

16. A demand function is given by: $P = 1500 - 3Q$ and the cost function is given by: $C = 12Q^2$. Find the value of Q at the equilibirum point.

    (a) 50

    (b) 100

    (c) 150

    (d) 125

17. A firm has a fixed production cost of ₹ 90 and a marginal variable production cost of ₹ 9. The price of the product is ₹18. Find the cost function, revenue function, and the value of Q at the Break Even point.

    (a) $2Q + 20$; $9Q$; 10

    (b) $9Q + 90$; $18Q$; 10

    (c) $4Q + 90$; $36Q$; 20

    (d) $Q + 10$; $5Q$; 50

18. A sugar industry has a fixed cost of ₹290 and a marginal cost of 50 paise. He sells sugar at the price of ₹15/kg. Find the quantity of sugar sold for breaking even.

    (a) 10

    (b) 30

    (c) 20

    (d) 40

19. A cotton mill has a fixed cost of ₹1540 and a marginal cost of ₹33. He sells a shirt at a price of ₹110. Find the minimum number of shirts sold so the mill incurs no loss.

    (a) 23

    (b) 20

    (c) 26

    (d) 29

20. Given: $C(x) = 900 + 30x + 0.6x^2$, $P = 90$. Find the value of x at equilibrium point.

    (a) 40

    (b) 50

    (c) 70

    (d) 30

21. Given: $C(x) = 2x^2 - 3x - 12$, P = 33. Find the value of x at equilibrium point.

(a) 9

(b) 12

(c) 15

(d) 18

22. Given: $R(x) = 3x^2 + 24x + 2$, $MC\,(x)$ = 42 and the fixed cost is 90. Find the value of x at equilibrium point.

(a) 2

(b) 3

(c) 7

(d) 11

23. Given: $R(x) = 6x^2 - 11x - 35$, $C(x) : 5x^2 - 3x + 16$. Find the value of x for which profit is maximum.

(a) 2

(b) 4

(c) 6

(d) 12

24. Given: $R(x) = 20x^2 - 15x - 10$, $C(x) : x^2 + 99x + 27$. Find the value of x for which profit is maximum.

(a) 3

(b) 15

(c) 25

(d) 10

25. Given: $R(x) = 3x^2 + 4x + 2$, $MC(x) = 16$ and the fixed cost is 24. Find the profit maximising value of x under perfect competition.

(a) 1

(b) 3

(c) 4

(d) 2

26. A manufacturer has a monthly fixed cost of ₹1, 00,000 and a production cost of ₹50 per unit produced. The product is sold at ₹75. Find the cost function and the number of products be sold by the manufacturerer to have break even.

(a) 25x + 50,000; 2000

(b) 50x + 1,00,000; 4000

(c) 5x + 1,00,000; 3000

(d) 2.5x + 10,000; 5000

27. A cement industry has a yearly fixed cost of ₹96,000 and a monthly production cost of ₹13 per unit produced. The product is sold at ₹39 per unit. Find the cost function

    (a) $13x + 8000$

    (b) $13x + 96{,}000$

    (c) $39x + 96{,}000$

    (d) $39x + 8000$

28. Find the monthly profit function if a firm's yearly fixed cost is ₹60,000 and yearly production cost is ₹120 per piece. Each unit is sold at ₹15.

    (a) $\pi = 5x - 5000$

    (b) $\pi(x) = 15x - 5000$

    (c) $\pi(x) = 20x - 5000$

    (d) $\pi(x) = 25x - 5000$

29. Given: $C(x) = 9x + 350$ and $P = 14$. Find the condition of getting break-even point

    (a) $5x - 350 = 0$

    (b) $7x - 350 = 0$

    (c) $-14x - 350 = 0$

    (d) None of these

30. With reference to Q. 29 find the break-even quantity

    (a) 50

    (b) 70

    (c) 110

    (d) 100

From Q.31 to Q.39, find whether the function is maximum or minimum:

31. $f(x) = 6x^2 + 11x - 35$

    (a) Maximum

    (b) Minimum

    (c) No Curvature

    (d) None of the above

32. $f(x) = -4x^2 - 7x - 35$
    (a) Maximum
    (b) Minimum
    (c) No Curvature
    (d) None of the above

33. $f(x) = 20x^2 - 15x - 10$
    (a) Maximum
    (b) Minimum
    (c) No Curvature
    (d) None of the above

34. $f(x) = 3x^2 - 4x + 2$
    (a) Maximum
    (b) Minimum
    (c) No Curvature
    (d) None of the above

35. $f(x) = -x^2 + 6x + 18$
    (a) Maximum
    (b) Minimum
    (c) No Curvature
    (d) None of the above

36. $f(x) = 9x^2 - 6x + 1$
    (a) Maximum
    (b) Minimum
    (c) No Curvature
    (d) None of the above

37. $f(x) = -x^2 + 4x - 2$

(a) Maximum

(b) Minimum

(c) No Curvature

(d) None of the above

38. $f(x) = 3x^2 + 2$

(a) Maximum

(b) Minimum

(c) No curvature

(d) None of the above

39. $f(x) = x^2 - 3x$

(a) Maximum

(b) Minimum

(c) No curvature

(d) None of the above

Find the values of *x* for which the functions of Q. No. 40 to 50 have maximum and minimum points

40. $f(x) = ax^3 + bx^2 + cx + d;\ a < 0;\ b < 0;\ c > 0\ and\ a < b$

(a) Maximum, $x = \dfrac{-b + \sqrt{b^2 - 3ac}}{3a}$; Minimum, $x = \dfrac{-b - \sqrt{b^2 - 3ac}}{3a}$

(b) Minimum, $x = \dfrac{-b + \sqrt{b^2 - 3ac}}{3a}$; Maximum, $x = \dfrac{-b - \sqrt{b^2 - 3ac}}{3a}$

(c) No Curvature

(d) None of the above

41. $f(x) = \dfrac{x^3}{3} - 9x^2 + 81x + 70$

(a) Maximum, $x = 9$; Minimum, $x = 9$

(b) Minimum, $x = 3$; Maximum, $x = 5$

(c) No curvature

(d) None of the above

42. $f(x)=\frac{2}{3}x^3+\frac{9}{2}x^2-11x-21$

(a) Maximum, $x = 2$; Minimum, $x=-\frac{9}{2}$

(b) Maximum, $x=-\frac{11}{2}$; Minimum, $x = 1$

(c) Maximum, $x=-\frac{3}{2}$; Minimum, $x = -3$

(d) No curvature

43. $f(x)=\frac{4}{3}x^3-5x^2+4x-9$

(a) Maximum, $x = 4$; Minimum, $x=-\frac{3}{2}$

(b) Maximum, $x=\frac{1}{2}$; Minimum, $x = 2$

(c) Maximum, $x = 2$; Minimum, $x = -1$

(d) No curvature

44. $f(x)=x^3-2x^2-4x$

(a) Maximum, $x=-\frac{2}{3}$; Minimum, $x = 2$

(b) Maximum, $x=\frac{3}{4}$; Minimum, $x = 1$

(c) Maximum, $x = 3$; Minimum, $x=-\frac{1}{3}$

(d) No curvature

45. $f(x) = \frac{x^3}{3} - 4.5x^2 + 8x + 2$

(a) Maximum, $x = 1$; Minimum, $x = 8$

(b) Maximum, $x = 4$; Minimum, $x = 2$

(c) Maximum, $x = 3$; Minimum, $x = -5$

(d) No curvature

46. $f(x)=\frac{x^3}{3}-\frac{3}{2}x^2+2x-3$

(a) Maximum, $x = 3$; Minimum, $x = 5$

(b) Maximum, $x = 1$; Minimum, $x = 2$

(c) Maximum, $x = 6$; Minimum, $x = 1$

(d) No curvature

47. $f(x) = \frac{2}{3}x^3 - \frac{3}{2}x^2 - 5x$

(a) Maximum, $x = -1$; Minimum, $x = \frac{5}{2}$

(b) Maximum, $x = 1$; Minimum, $x = 3$

(c) Maximum, $x = -1$; Minimum, $x = -\frac{3}{2}$

(d) No curvature

48. $f(x) = -\frac{x^3}{3} + 4x^2 - 15x$

(a) Maximum, $x = 1$; Minimum, $x = 3$

(b) Maximum, $x = 5$; Minimum, $x = 3$

(c) Maximum, $x = -3$; Minimum, $x = -5$

(d) No curvature

49. $f(x) = \frac{x^3}{3} - \frac{x^2}{2} - 2x$

(a) Maximum, $x = -1$; Minimum, $x = 2$

(b) Maximum, $x = 1$; Minimum, $x = -2$

(c) Maximum, $x = -3$; Minimum, $x = 5$

(d) No curvature

50. $f(x) = \frac{x^3}{3} + 2x^2 + 3x + 7$

(a) Maximum, $x = -3$; Minimum, $x = -1$

(b) Maximum, $x = \frac{1}{2}$; Minimum, $x = 2$

(c) Maximum, $x = 1$; Minimum, $x = 4$

(d) No curvature

Solution:

- **Multiple Choice Questions (MCQ)**

1	(c)	Put $x = h + 3$ and then evaluate
2	(c)	Put $x = 0$ and evaluate
3	(c)	Use the formulae $a^2 - b^2 = (a+b)(a-b)$ and then put $x = 4$
4	(b)	By using the formulae
5	(d)	By using the formulae
6	(c)	Use logarithm on both sides
7	(a)	By using the formulae
8	(d)	By using the formulae
9	(b)	By using the formulae
10	(c)	By using the formulae
11	(a)	$MR = \frac{dR}{dQ}$ *and* $MC = \frac{dC}{dQ}$, then Put $MR = MC$
12	(b)	$MR = \frac{dR}{dQ}$ *and* $MC = \frac{dC}{dQ}$, then Put $MR = MC$
13	(c)	$MR = \frac{dR}{dQ}$ *and* $MC = \frac{dC}{dQ}$, then Put $MR = MC$
14	(b)	$MR = \frac{dR}{dQ}$ *and* $MC = \frac{dC}{dQ}$, then Put $MR = MC$
15	(c)	$MR = \frac{dR}{dQ}$ *and* $MC = \frac{dC}{dQ}$, then Put $MR = MC$
16	(a)	$MR = \frac{dR}{dQ}$ *and* $MC = \frac{dC}{dQ}$, then Put $MR = MC$
17	(b)	Profit Function = Revenue Function – Cost Function Put profit function = 0
18	(c)	Profit Function = Revenue Function – Cost Function Put profit function = 0

19	(b)	Profit Function = Revenue Function – Cost Function Put profit function = 0
20	(b)	$MR = \frac{dR}{dx}$ *and* $MC = \frac{dC}{dx}$ then, Put $MR = MC$
21	(a)	$MR = \frac{dR}{dx}$ *and* $MC = \frac{dC}{dx}$ then, Put $MR = MC$
22	(b)	$MR = \frac{dR}{dx}$ *and* $MC = \frac{dC}{dx}$ then, Put $MR = MC$
23	(b)	$MR = \frac{dR}{dx}$ *and* $MC = \frac{dC}{dx}$ then, Put $MR = MC$
24	(a)	$MR = \frac{dR}{dx}$ *and* $MC = \frac{dC}{dx}$ then, Put $MR = MC$
25	(d)	$MR = \frac{dR}{dx}$ *and* $MC = \frac{dC}{dx}$ then, Put $MR = MC$
26	(b)	Profit Function = Revenue Function – Cost Function Put profit function = 0
27	(b)	Total Cost = Fixed Cost + Variable Cost (Production cost)
28	(a)	First divide the production cost by 12 in order to find out the monthly production cost. Then, profit function = Revenue function – cost function
29	(a)	Revenue Function = Price × Quantity. So, $R(x) = P \times x$ Then Profit Function = Revenue Function – Cost Function = 0
30	(b)	Put Profit Function = 0
31	(b)	Find $f''(x)$ Then, if $f''(x) > 0 \rightarrow$ Minimum if $f''(x) < 0 \rightarrow$ Maximum
32	(a)	Find $f''(x)$ Then, if $f''(x) > 0 \rightarrow$ Minimum if $f''(x) < 0 \rightarrow$ Maximum

33	(b)	Find $f''(x)$ Then, if $f''(x) > 0 \rightarrow$ Minimum if $f''(x) < 0 \rightarrow$ Maximum
34	(b)	Find $f''(x)$ Then, if $f''(x) > 0 \rightarrow$ Minimum if $f''(x) < 0 \rightarrow$ Maximum
35	(a)	Find $f''(x)$ Then, if $f''(x) > 0 \rightarrow$ Minimum if $f''(x) < 0 \rightarrow$ Maximum
36	(b)	Find $f''(x)$ Then, if $f''(x) > 0 \rightarrow$ Minimum if $f''(x) < 0 \rightarrow$ Maximum
37	(a)	Find $f''(x)$ Then, if $f''(x) > 0 \rightarrow$ Minimum if $f''(x) < 0 \rightarrow$ Maximum
38	(b)	Find $f''(x)$ Then, if $f''(x) > 0 \rightarrow$ Minimum if $f''(x) < 0 \rightarrow$ Maximum
39	(b)	Find $f''(x)$ Then, if $f''(x) > 0 \rightarrow$ Minimum if $f''(x) < 0 \rightarrow$ Maximum
40	(a)	Find $f'(x)$ and put $f'(x) = 0$ to find out the roots. Now find $f''(x)$ and put the roots in $f''(x)$ in order to find out the value of $f''(x)$ Then, if $f''(x) > 0 \rightarrow$ Minimum if $f''(x) < 0 \rightarrow$ Maximum

41	(c)	Find $f'(x)$ and put $f'(x)=0$ to find out the roots. Now find $f''(x)$ and put the roots in $f''(x)$ in order to find out the value of $f''(x)$ Then, if $f''(x)>0 \rightarrow$ Minimum if $f''(x)<0 \rightarrow$ Maximum and if $f''(x)=0 \rightarrow$ No curveture
42	(b)	Find $f'(x)$ and put $f'(x)=0$ to find out the roots. Now find $f''(x)$ and put the roots in $f''(x)$ in order to find out the value of $f''(x)$ Then, if $f''(x)>0 \rightarrow$ Minimum if $f''(x)<0 \rightarrow$ Maximum
43	(b)	Find $f'(x)$ and put $f'(x)=0$ to find out the roots. Now find $f''(x)$ and put the roots in $f''(x)$ in order to find out the value of $f''(x)$ Then, if $f''(x)>0 \rightarrow$ Minimum if $f''(x)<0 \rightarrow$ Maximum
44	(a)	Find $f'(x)$ and put $f'(x)=0$ to find out the roots. Now find $f''(x)$ and put the roots in $f''(x)$ in order to find out the value of $f''(x)$ Then, if $f''(x)>0 \rightarrow$ Minimum if $f''(x)<0 \rightarrow$ Maximum
45	(a)	Find $f'(x)$ and put $f'(x)=0$ to find out the roots. Now find $f''(x)$ and put the roots in $f''(x)$ in order to find out the value of $f''(x)$ Then, if $f''(x)>0 \rightarrow$ Minimum if $f''(x)<0 \rightarrow$ Maximum
46	(b)	Find $f'(x)$ and put $f'(x)=0$ to find out the roots. Now find $f''(x)$ and put the roots in $f''(x)$ in order to find out the value of $f''(x)$ Then, if $f''(x)>0 \rightarrow$ Minimum if $f''(x)<0 \rightarrow$ Maximum
47	(a)	Find $f'(x)$ and put $f'(x)=0$ to find out the roots. Now find $f''(x)$ and put the roots in $f''(x)$ in order to find out the value of $f''(x)$ Then, if $f''(x)>0 \rightarrow$ Minimum if $f''(x)<0 \rightarrow$ Maximum

48	(b)	Find $f'(x)$ and put $f'(x)=0$ to find out the roots. Now find $f''(x)$ and put the roots in $f''(x)$ in order to find out the value of $f''(x)$ Then, if $f''(x)>0 \rightarrow$ Minimum if $f''(x)<0 \rightarrow$ Maximum
49	(a)	Find $f'(x)$ and put $f'(x)=0$ to find out the roots. Now find $f''(x)$ and put the roots in $f''(x)$ in order to find out the value of $f''(x)$ Then, if $f''(x)>0 \rightarrow$ Minimum if $f''(x)<0 \rightarrow$ Maximum
50	(a)	Find $f'(x)$ and put $f'(x)=0$ to find out the roots. Now find $f''(x)$ and put the roots in $f''(x)$ in order to find out the value of $f''(x)$ Then, if $f''(x)>0 \rightarrow$ Minimum If $f''(x)<0 \rightarrow$ Maximum